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A 5d/2d/4d correspondence

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Abstract:

We propose a correspondence between two-dimensional $(0,4)$ sigma models with target space the moduli spaces of r monopoles, and four-dimensional $\mathcal{N}=4$, $U(r)$ Yang-Mills theory on del Pezzo surfaces. In particular, the two- and four-dimensional BPS partition functions are argued to be equal. The correspondence relies on insights from five-dimensional supersymmetric gauge theory and its geometric engineering in M-theory, hence the name "5d/2d/4d correspondence". We provide various tests of our proposal. The most stringent ones are for $r=1$, for which we prove the equality of partition functions.

Contents

1 Introduction and summary

The main aim of this paper is to present a new correspondence between two seemingly different objects, namely the elliptic genus of magnetic monopole moduli spaces and the partition function of $\mathcal{N}=4$ super Yang-Mills theory on del Pezzo surfaces. It is remarkable that there exists any kind of a relation between magnetic monopole moduli spaces and del Pezzo surfaces, as these spaces have nothing to do with each other at first instance. The statement that an elliptic genus of a 2d CFT is computed by a partition function of a 4d gauge theory hints towards an explanation in terms of M5-branes wrapped over different manifolds of different dimensions. Our introduction and motivation will therefore start with a short discussion on M5-branes and their partition functions, after which we propose the correspondence that leads to the main conjecture in formula (1.1). At the end, in Section (1.3), we give a first explanation of the conjecture using 5d $\mathcal{N}=1$ supersymmetric gauge theories.

1.1 Motivation: M5-branes

Understanding the M5-brane worldvolume theory and formulating a consistent action for this theory has been a long standing open problem. First steps towards the solution were taken in [1]. One of the main difficulties in the description is the existence of a self-dual three-form H, which is the field strength of the five-brane two-form B, and for which no Lagrangian formulation is available. This fact also makes it difficult to define a convenient partition function for the M5-brane theory. However, in [1] it was noticed that the partition function is a certain section of a line bundle over the intermediate Jacobian $J_W = H^3(W, \mathbb{R})/H^3(W, \mathbb{Z})$ where W is the six-manifold on which the five-brane is wrapped. This can be traced back to the coupling of the two-form B to the M-theory three-form C and can be understood in an intuitive manner as follows. Consider the specific case of $W = \Sigma \times \mathbb{P}^2$, with Σ a Riemann surface whose characteristic length is much larger than that of $\mathbb{P}^2$. Then the theory of the chiral two-form reduces to a chiral scalar on Σ. This chiral scalar can now be coupled to a $U(1)$ gauge field A which when setting the curvature $F = dA$ to zero defines modulo gauge transformations a point on $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$. In order for the gauge field to couple to the chiral part of the scalar only, the Lagrangian contains a term which breaks gauge invariance. Thus the partition function $\mathcal{Z}$, defined by taking the path integral over the scalar, is not a function of A but rather a section of a line bundle over $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$. This point of view is convenient when one wants to establish contact with wave functions and a background independent interpretation of partition functions [2]. It also naturally leads to a description of the partition function in terms of theta-functions which can be interpreted as sections of line bundles on the Jacobian.

A more general situation where one considers instead of $\mathbb{P}^2$ an arbitrary Kähler manifold P and takes Σ to be the torus T^2 has been extensively analyzed in [3–6]. In this case the reduction of the two-form will give rise to left-moving and right-moving chiral scalars whose numbers are determined by the self-dual and anti-self-dual harmonic two-forms on P. One then considers a situation where P is embedded into a Calabi-Yau threefold X and takes the embedding to be holomorphic in order to preserve supersymmetry. The deformation degrees of freedom of the five-brane together with the chiral scalars from the reduction of the two-form form a $(0, 4)$ CFT whose partition function is a modular form with its modular parameter being the complex structure τ of T^2. In [4] it is argued that there is an underlying sigma model for this CFT whose target space $\mathcal{E}$ is a bundle of the form $\mathcal{V} \to \mathcal{M}$ where $\mathcal{V}$ is a vector-bundle with real rank $b_2^-(P) - b_2^+(P) - (h^{1,1}(X) - 2)$ and the base-manifold $\mathcal{M}$ carries a hyperkähler structure. One drawback of present day constructions of the bundle $\mathcal{E}$ is that they are only known for the case of a single M5-brane. Thus a major question to answer is what happens if one wraps an arbitrary number of five-branes around P. We will provide an answer to this question when P is a del Pezzo surface and X is non-compact.

Yet another viewpoint is obtained by turning the situation around. Instead of taking the size of P to be small one considers a limit where the characteristic length of P is much larger than that of T^2. Reduction of the degrees of freedom of r five-branes to P leads

to topologically twisted $\mathcal{N} = 4$ supersymmetric $U(r)$ Yang-Mills theory [7, 8]. The key point here is that the complex structure τ of T^2, will take over the role of the complexified gauge coupling of the Yang-Mills theory. The $U(r)$ partition functions can be evaluated in various cases. One example is the case where P is an elliptic fibration, such that one can invoke twice T-duality along the elliptic fiber to map the Yang-Mills partition function for arbitrary rank r to topological string free energies [8–10]. More generally, Yang-Mills partition functions for arbitrary rank r can be evaluated for all rational surfaces using algebraic-geometric techniques as wall-crossing and blow-up formulas [11–16].

1.2 Statement of the conjecture

The two theories presented above, namely the $(0,4)$ CFT on T^2 and the $\mathcal{N} = 4$ SYM on P, are related by the effective action of the M5-brane world-volume theory. Indeed, at the level of partition functions one can consider the elliptic genus of the two-dimensional theory and a particular topologically twisted version of Yang-Mills theory[1]. Both quantities will be index-like and therefore their dependence on the volumes of T^2 and P enters only in a trivial way. Moreover, both quantities are modular functions, with equal modular weight $(0,2)$. This suggests that the underlying six-dimensional theory will ultimately connect the two and one expects a relation of the following form

$$\mathcal{Z}^{(r)}_{(0,4)\ \mathrm{CFT}} = \mathcal{Z}^{(r)}_{\mathrm{SYM}}\ , \tag{1.1}$$

where r denotes the number of M5-branes wrapped around $T^2 \times P$.

The correspondence (1.1) is reminiscent to the one studied in [17]. But note that in the present case both 2d and 4d theories are supersymmetric whereas in [17] the dual 2d theory is nonsupersymmetric Liouville theory. The task of this paper is to shed light on the correspondence (1.1) and in particular clarify the nature of the underlying sigma model which gives rise to the $(0,4)$ CFT. To do this, we restrict to the case where P is a del Pezzo surface and state a conjecture for (1.1) for arbitrary high r.

Del Pezzo surfaces are Kähler manifolds of complex dimension two with positive anti-canonical class, which makes them rigid inside the Calabi-Yau. Such surfaces are either $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$ or blow-ups of these. Note that blowing-up a point of $\mathbb{F}_0$ or $\mathbb{F}_1$ give topologically equal manifolds. We define $\mathbb{B}_0 := \mathbb{F}_0$ and $\mathbb{B}_n$ as the blow-up of $\mathbb{B}_0$ at n points. We will not consider in detail the case $P = \mathbb{F}_1$, since the all qualitative aspects of the correspondence can be explained using the class $\mathbb{B}_n$. For these surfaces $b_2^+(P) = 1$, such that interesting wall-crossing effects arise in the $\mathcal{N} = 4$ Yang-Mills theory on P.

Let us elaborate further on the formulation of our conjecture. It says that wrapping r M5-branes around $T^2 \times \mathbb{B}_n$ and taking the size of T^2 to be much larger than that of $\mathbb{B}_n$ will give rise to a $(0,4)$ sigma model with target space being the moduli space of r $SU(2)$ magnetic monopoles in the presence of $2rn$ fermionic zero modes coming from adding massless flavor fermions in the fundamental representation to the $SU(2)$ gauge theory. Hence the number of blow-ups is identified with the number of flavors, $N_f = n$. The

[1] As will be discussed in Section 3, the reduction of the five-brane theory along T^2 will automatically lead to a topological theory on P.

structure of the target space of the $(0,4)$ sigma model has the following bundle structure [18]:

$$\begin{array}{ccc} O(r)\times SO(2N_f) & & \\ \downarrow & & \\ \mathcal{M}_r & = & \mathbb{R}^3\times\frac{S^1\times\widetilde{\mathcal{M}}_r}{\mathbb{Z}_r}\ . \end{array} \tag{1.2}$$

In the above, $\mathcal{M}_r$ denotes the moduli space of magnetic monopoles of charge r in pure $SU(2)$ Yang-Mills theory. The factor $\mathbb{R}^3$ can be understood as the zero modes corresponding to the center of mass motion in space, S^1 represents the unbroken $U(1)$ charge of the monopoles and $\widetilde{\mathcal{M}}_r$ captures the relative moduli space. Over the base $\mathcal{M}_r$, there is a $O(r)$ vector bundle with transition functions in the orthogonal group $O(r)$. This bundle is also called the Index bundle associated to the magnetic monopole moduli space, as discussed in [18]. Finally, there is an additional isometric action of the flavor group $SO(2N_f)$ which we discuss in the following sections below.

We now claim that the elliptic genus, as defined in Section 4, of the sigma model corresponding to this target space is equal to the partition function of topologically twisted $U(r)$ $\mathcal{N}=4$ SYM on $\mathbb{B}_{n=N_f}$. The topological twist is the one introduced in [7] and as discussed there the path-integral of the twisted Yang-Mills theory localizes on instanton configurations.

1.3 Explanation of the conjecture: a 5d/2d/4d correspondence

We turn to an explanation of the main idea behind the claimed correspondence. Consider five-dimensional $\mathcal{N}=1$ supersymmetric gauge theory with $SU(2)$ gauge group whose properties have been first discussed in [19]. This theory can be geometrically engineered, as was done in [20, 21] (see also [22]), by compactifying M-theory on a Calabi-Yau threefold which is locally the canonical bundle over the del Pezzo surface $\mathbb{B}_{N_f}$. Here, N_f denotes the number of flavors in the fundamental of $SU(2)$. These are accompanied with a $U(1)^{N_f}$ flavor-symmetry which gets enhanced to $SO(2N_f)$ in the massless case. The spectrum of the five-dimensional gauge theory contains among the W-bosons also instanton particles[2] and the magnetic string. This instantonic particle is also called the dyonic instanton, as it has both instanton number and electric charge [23]. The magnetic string is the uplift of the magnetic monopole in four-dimensions and has therefore locally the same moduli space as its four-dimensional companion. The worldvolume dynamics of this string is a $(0,4)$ CFT with the magnetic monopole moduli space as a target. From the point of view of geometric engineering, the string arises from compactifying an M5-brane on the del Pezzo $\mathbb{B}_{N_f}$.

The next step is to compactify the five-dimensional theory on T^2 down to three-dimensional $\mathcal{N}=4$ SYM [22]. This way the magnetic string wrapped on T^2 becomes an instanton in the three-dimensional theory. Computing such an instanton contribution amounts to performing a certain path-integral over the instanton moduli space. But in our case this is just the moduli space of magnetic monopoles. On the other hand, from the viewpoint of the M-theory setup the path integral is equivalent to computing the partition function of the M5-brane, or, after compactifying on T^2 the partition function of

[2]These become the usual instantons of the four-dimensional gauge theory once compactifying on S^1.

(topologically twisted) $\mathcal{N}=4$ SYM on the del Pezzo. This immediately opens the door to what was said in the previous section.

1.4 Outline

In Section 2 we start by describing the worldvolume theory of the two-dimensional magnetic string from the viewpoint of five-dimensional supersymmetric gauge theory. The metric on the moduli space captures the dynamics of the magnetic string and shall be reviewed in some detail before turning to the description of the action and conformal field theory. In Section 3 we then proceed to a presentation of the geometric engineering picture which contains a description of the Coulomb branch in terms of moduli of the Calabi-Yau. This way it is possible to make contact with the M5-brane and the gauge theory on the del Pezzo surface. Finally, in Section 4 we formulate and provide tests of the conjecture. For magnetic charge $r=1$, we explicitly compute both elliptic genus and the partition function of the $\mathcal{N}=4$ SYM on del Pezzo surfaces, and show they are the same. Furthermore, we provide predictions for the elliptic genus of the CFT for $r=2$ by computing the partition function on the 4d side. By compactifying the 5d gauge theory on a circle, we also make the connection to the BPS states of four-dimensional $\mathcal{N}=2$ SYM providing further insight and evidence for the conjecture. We end the paper with a short conclusion and outlook in Section 5.

2 $(0,4)$ CFT's from 5d supersymmetric gauge theories

Throughout this section, we consider five-dimensional $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $G=SU(2)$ and $N_f \leq 8$ massless flavors in the fundamental representation [19]. The extension to massive hypermultiplets is considered at the end of this section. On the Coulomb branch of five-dimensional supersymmetric gauge theories, the spectrum contains a magnetic string as a solitonic BPS configuration. It can best be understood as the uplift of a BPS magnetic monopole in four dimensions. The gauge group $SU(2)$ is broken to $U(1)$, and we denote the vacuum expectation value of the real adjoint scalar in five dimensions by ϕ. Using a Weyl-reflection, we can always assume it to be positive.

The tension T of the BPS magnetic string is given by[3]

$$\frac{T}{\sqrt{2}} = r\Big(\frac{\phi}{2g_5^2} + \frac{\kappa}{4}\phi^2\Big) \ , \tag{2.1}$$

where g_5 is the bare (dimensionfull) five-dimensional Yang-Mills coupling constant and κ determines the one-loop correction to the effective coupling constant. The factor κ is also the coefficient in front of the one-loop induced Chern-Simons term in five dimensions [19], which we assumed to be absent in the classical, microscopic theory. For N_f massless flavors, we have

$$\kappa = 2(8-N_f) \ . \tag{2.2}$$

[3]Compared to [22], our formula contains a factor of one-half. As $\frac{T}{\sqrt{2}} = r\phi_D = r\frac{\partial \mathcal{F}}{\partial \phi}$, one can think of this as a rescaling of the prepotential $\mathcal{F}$ by $\frac{1}{2}$. This is done to suppress factors of two in later sections.

The requirement $N_f \leq 8$ alluded to in the beginning of this section, is reflected by the fact that the magnetic string must have positive tension (we always assumed $\phi > 0$ without loss of generality). For $N_f > 8$ this tension becomes negative for some values of ϕ and moreover, the metric on the Coulomb branch of the five-dimensional theory becomes singular [19]. Hence we do not consider $N_f > 8$.

The spectrum also contains another solitonic BPS state, namely the dyonic instanton [23]. It is the uplift of a four-dimensional instanton to a point particle in five dimensions and is therefore classified by an instanton charge n_I. For charge $n_I = 1$, the dyonic instanton contributes to the mass, or central charge[4]

$$Z_I = \frac{1}{2g_5^2} + \frac{1}{2}\kappa\,\phi\ , \tag{2.3}$$

where the second term is due to the one-loop induced Chern-Simons term. The dyonic instanton also acquires an electric charge n_e in five dimensions. The total central charge is then given by

$$Z = n_e\phi + n_I Z_I\ . \tag{2.4}$$

In this section, we show how a two-dimensional $(0,4)$ CFT emerges from the dynamics of collective coordinates of the magnetic string. As we compactify a single-charged magnetic string on a circle, its momentum and winding modes are related to dyonic instanton charges n_e and n_I, as we will show. This is similar to how a magnetic monopole can acquire electric charge and become dyonic. Furthermore we can consider the magnetic string in five-dimensional supersymmetric gauge theory on $\mathbb{R}^3 \times T^2$. When the worldsheet of the string wraps the torus T^2, it manifests itself as an instanton in $\mathbb{R}^3$. Such an instanton corrects the metric on the Coulomb-branch of the three-dimensional low-energy effective action through an instanton induced four-fermion correlator, see [24, 25].

2.1 Magnetic string dynamics and quantization

Before we discuss the dynamics of the magnetic string, we first review some well-known aspects of magnetic monopole dynamics. They are BPS objects in four-dimensional supersymmetric gauge theories, which we think of as the zero radius limit of the five-dimensional gauge theory on $\mathbb{R}^4 \times S^1$. For some background material on BPS magnetic monopoles, see e.g. [26, 27]. Our strategy to obtain the $(0,4)$ CFT, is to lift the supersymmetric quantum mechanics of the monopole to a two-dimensional sigma-model defined on the worldsheet of the magnetic string. These sigma models have also been derived in [28] in an even more general setting, but without the link to 5d supersymmetric gauge theories. Moreover, we are interested in the properties of the corresponding conformal field theory, as we discuss below.

Consider a static magnetic monopole of charge r satisfying the Bogomol'nyi equations on $\mathbb{R}^3$. For gauge group $G = SU(2)$, such a solution is parametrized by $4r$ bosonic collective

[4]Again, compared to [22] , our central charge formula contains a factor of one-half. This is consistent with formula (2.1) as the instanton charge is given by $Z_I = \frac{\partial^2 \mathcal{F}}{\partial \phi^2}$.

coordinates on the moduli space

$$\mathcal{M}_r = \mathbb{R}^3 \times \frac{S^1 \times \widetilde{\mathcal{M}}_r}{\mathbb{Z}_r} \ . \tag{2.5}$$

For $r = 1$, the moduli space is just the universal factor

$$\mathcal{M}_1 = \mathbb{R}^3 \times S^1 \ , \tag{2.6}$$

parametrized by the three positions of the monopole $\vec{X} \in \mathbb{R}^3$ and a gauge orientation zero mode which is an angle denoted by $\theta \in [0, 2\pi]$.

In general, both $\mathcal{M}_r$ and $\widetilde{\mathcal{M}}_r$ are hyperkähler manifolds, and e.g. $\widetilde{\mathcal{M}}_2$ is (the double cover of) the Atiyah-Hitchin manifold [29]. The dimension of the relative moduli space $\widetilde{\mathcal{M}}_r$, obtained after taking out the center of mass coordinate, is $4(r-1)$. The discrete symmetry $\mathbb{Z}_r$ acts on both the S^1 as on $\widetilde{\mathcal{M}}_r$. Its precise action is of not much importance for our analysis, hence we refrain from giving its definition. The group of continuous isometries of the moduli space $\mathcal{M}_r$ is the product of Euclidean group in three dimensions and a phase rotation,

$$G_{isom} = SO(3) \ltimes \mathbb{R}^3 \times U(1) \ . \tag{2.7}$$

The translation group $\mathbb{R}^3$ simply acts as translations on the center of mass coordinates $\vec{X}$. The rotations also act as rotations on $\vec{X}$, but its action on $\widetilde{\mathcal{M}}_r$ is more complicated in general. For the Atiyah-Hitchin manifold $\widetilde{\mathcal{M}}_2$, it is known that there is an $SO(3)$ group of isometries which rotates the complex structures on $\widetilde{\mathcal{M}}_2$ [29]. Finally, the $U(1)$ factor arises from the large gauge transformations. On the moduli space, it acts by rotation on the S^1.

In the fermionic sector, we have $4r$ zero modes coming from the adjoint fermions in the vector multiplet and $2rN_f$ from the fermions in the hypermultiplets. These zero modes are solutions of the massless three-dimensional Dirac equation in the presence of the magnetic monopole, and their number can be computed using Callias index theorems [30]. We parametrize the fermionic zero modes by Grassmann odd collective coordinates

$$\psi^m \ , \quad m = 1, ..., 4r \ , \qquad \chi^A \ , \quad A = 1, ..., 2rN_f \ . \tag{2.8}$$

When masses for the hypermultiplets are present, the counting of the fermionic zero modes changes. We discuss this in the next subsection.

In the semiclassical approach of soliton quantization, one is interested in the fluctuations around constant (static) configurations, i.e. in the dynamics of the solitonic particle. To study this, we let all the collective coordinates and fermionic zero modes depend on time, and study the motion of the monopole in the low-energy approximation (i.e. for small velocities). This leads to a formulation in terms of a supersymmetric quantum mechanics on the moduli space with action given by [31–33],

$$S_{QM} = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t \left[g_{mn}\left(\dot{X}^m \dot{X}^n + i\psi^m D_t \psi^n\right) + i\chi^A D_t \chi^A - \frac{1}{2} F_{mnAB}\, \psi^m \psi^n \chi^A \chi^B \right] \ , \tag{2.9}$$

where g_{mn} is the metric on the moduli space $\mathcal{M}_r$ with coordinates $X^m; m = 1, ..., 4r$. The supersymmetric quantum mechanics defined by (2.9) has four supercharges. These are the

four supercharges that are unbroken by the BPS magnetic monopole. Four supercharges require the metric g on $\mathcal{M}_r$ to be hyperkähler, which is known to be the case. The covariant derivatives in (2.9) are defined as

$$D_t\psi^m = \dot{\psi}^m + \dot{X}^n\Gamma^m{}_{np}\psi^p \ , \qquad D_t\chi^A = \dot{\chi}^A + \dot{X}^m\omega^A_{m\,B}\chi^B \ , \tag{2.10}$$

where Γ is the Levi-Cevita connection on $\mathcal{M}_r$ and ω is a connection on the $\mathcal{O}(r)$ index bundle whose curvature is F_{mnAB} [18].

For $r = 1$ the moduli space is given by the universal factor (2.6). The action for the supersymmetric quantum mechanics for $r = 1$ becomes free,

$$S^{r=1}_{QM} = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t \left[g_{mn}\left(\dot{X}^m\dot{X}^n + i\psi^m\dot{\psi}^n\right) + i\chi^A\dot{\chi}^A\right] \ , \tag{2.11}$$

with metric given by $g_{mn}\dot{X}^m\dot{X}^n = M|\dot{\vec{X}}|^2 + (M/|\phi|^2)\,\dot{\theta}^2$, where ϕ is the vev of the scalar field, in the notation of [22], and M is the monopole mass. Finally, $\theta \in [0, 2\pi]$ is a global $U(1)$ charge angle that parametrizes the circle S^1 in the moduli space. The radius of this circle is given by $R = 1/\phi$ in our conventions. The gauge transformation $-\mathrm{diag}(1, 1) \in SU(2)$ leaves invariant the gauge and Higgs field, but acts on the fermionic zero modes by -1. Therefore, the $O(1) = \{1, -1\}$ index bundle is the Möbius bundle [18], i.e. the flavor fermions change sign after making a half-period around the S^1:

$$\theta \to \theta + \pi, \qquad \chi^A \to -\chi^A. \tag{2.12}$$

This transformation must leave invariant the spectrum of the supersymmetric quantum mechanics. Leaving out for notational simplicity the fermions ψ^m, which are invariant under (2.12), then a general ground-state in the quantized Hilbert space of the quantum mechanics (2.11) is given by

$$|\Psi\rangle = e^{in_e\theta}e^{i\vec{k}\cdot\vec{X}} \otimes \chi^{A_1}\cdots\chi^{A_n}|0\rangle, \tag{2.13}$$

where $\vec{k}$ is the momentum along the non-compact directions $\vec{X} = \{X^1, X^2, X^3\}$ and n_e is the momentum along the compact S^1. A nonzero momentum along S^1 makes the monopole dyonic with electric charge n_e. Furthermore, the χ^A are viewed as operators satisfying the Clifford-algebra

$$\{\chi^A, \chi^B\} = \delta^{AB}. \tag{2.14}$$

Under the transformation (2.12) this state changes as follows

$$|\Psi\rangle \mapsto e^{in_e(\theta+\pi)}e^{i\vec{k}\cdot\vec{X}} \otimes (-\chi^{A_1})\cdots(-\chi^{A_n})|0\rangle = (-)^{n_e}(-)^H|\Psi\rangle, \tag{2.15}$$

where use has been made of the fact that the chirality operator $(-)^H$ anti-commutes with the χ^A for $A = 1, \cdots, 2N_f$. As (2.12) must be a symmetry of the quantum mechanics we arrive at the constraint [33, 34]

$$(-)^{n_e}|\Psi\rangle = (-)^H|\Psi\rangle. \tag{2.16}$$

This constraint implies a correlation between the electric-magnetic and flavor charges of the four-dimensional gauge theory, which avoids the unphysical decay of the BPS particles [34], for example decay of the W-boson into two particles of electric charge one.

It is now easy to uplift the quantum mechanics to a two-dimensional sigma model. The monopole becomes a string in five dimensions, and the supersymmetric quantum mechanics becomes a non-linear two-dimensional sigma model. The uplift is done in such a way that it preserves four supercharges, since the magnetic string is BPS in five dimensions. Therefore, its worldsheet dynamics must preserve either $(2,2)$ or $(0,4)$ supersymmetry. However, the multiplet structure of such models imposes strong constraints, and only $(0,4)$ supersymmetry is possible. The reason is that the χ^A do not have bosonic superpartners, whereas ψ^m sit in the same multiplet as the bosonic coordinates X^m. As a consequence, the χ^A must be part of the left-moving sector, and ψ^m sits in the right-moving supersymmetric sector.

The action for the $(0,4)$ model is,

$$S = \tilde{T} \int_{T^2} \mathrm{d}^2\sigma \Big[g_{mn} \Big(\partial_+ X^m \partial_- X^n + i\psi^m D_+ \psi^n \Big) + i\chi^A D_- \chi^A - \frac{1}{2} F_{mnAB}\, \psi^m \psi^n \chi^A \chi^B \Big] \ , \tag{2.17}$$

where we introduced coordinates on the worldsheet $\sigma^\pm$ and defined

$$D_+\psi^m = \partial_+\psi^m + \partial_+ X^n \Gamma^m{}_{np}\psi^p \ , \qquad D_-\chi^A = \partial_-\chi^A + \partial_- X^m \omega^A_{m\,B}\chi^B \ . \tag{2.18}$$

The tension $\tilde{T}$ will be proportional to the tension T in (2.1) by a numerical factor that we determine below. For later purposes, to define the elliptic genus, we have defined the $(0,4)$ model on a two-torus T^2. The magnetic string then wraps the T^2 and becomes an instanton in the three-dimensional gauge theory.

The action (2.17) falls into the class of $(0,4)$ supersymmetric sigma models [35–37]. The most general Lagrangian with $(0,4)$ also contains mass terms and a b-field, but they do not appear in our setup. In that case, $(0,4)$ supersymmetry implies the target space to be a hyperkähler manifold $(\mathcal{M}, g)$ equipped with a vector bundle E with connection ω. In the absence of masses or scalar potential on $\mathcal{M}$, the action defines a conformal field theory. It was argued in [38] that conformality can be maintained at the quantum level[5]. The central charges in the left and right-moving sectors can be computed in the limit of zero coupling constant, where the interaction between the monopoles vanishes. Using this argument, the calculation of the central charges is that of free fields for which we obtain

$$c_L = r(4+N_f) \ , \qquad c_R = 6r \ . \tag{2.19}$$

For magnetic charge $r=1$, the $(0,4)$ model becomes free, since the curvature of the $O(1)$ bundle vanishes. The resulting action is that of a free $(0,4)$ CFT with action

$$S = \tilde{T} \int_{T^2} \mathrm{d}^2\sigma \Big[g_{mn} \Big(\partial_+ X^m \partial_- X^n + i\psi^m \partial_+ \psi^n \Big) + i\chi^A \partial_- \chi^A \Big] \ , \tag{2.20}$$

[5] See also [39, 40] for related discussions. If the $(0,4)$ sigma model we consider is not conformal at the quantum level, then we assume that it flows to some fixed point in the infrared. The cases we discuss explicitly in this paper, namely the ones with $r=1$ are (orbifolds of) free field theories and hence are true CFT's.

containing three non-compact bosons $\vec{X}$, one compact boson θ. In the right-moving sector, there are four fermionic superpartners, and in the left-moving sector there is a flavor symmetry group $SO(2N_f)$ rotating the fermions $\chi^A; A = 1, ..., 2N_f$. The metric components g_{mn} are constant in this basis and given in the text after (2.11).

Upon uplifting the quantum mechanics for the monopole to the 2d CFT of the magnetic string one has to take into account the fundamental constraint (2.16). This is done as follows. On the one hand, one can see from the mode expansion of the left-moving fermions

$$\chi^A = \sum_n \rho_n^A e^{-n(t+ix)} \ , \tag{2.21}$$

that the action $\chi^A \mapsto -\chi^A$ translates to

$$\rho_n^A \mapsto -\rho_n^A \ , \tag{2.22}$$

which induces the natural generalization of the chirality-operator H to the fermion number operator $(-)^F$ acting on physical states. Now states in the Hilbert space of the magnetic string have to satisfy the constraint

$$(-)^{n_e+F}|\theta\rangle \otimes |\chi\rangle = |\theta\rangle \otimes |\chi\rangle, \tag{2.23}$$

where $|\theta\rangle$ denotes a state in the Hilbert space of the compact boson θ and $|\chi\rangle$ is obtained by quantizing the fermions χ^A. But this is nothing else than an orbifold of a 2d CFT by a group action $G = \{1, g\}$ generated by the identity and $g = (-)^{n_e+F}$.

On the other hand, unlike in quantum mechanics, the momentum of the magnetic string along S^1 is no longer simply identified with the electric charge n_e. The identification of the electric charge is rather deduced from the excitation modes of the magnetic string and their relation to BPS states of the gauge theory. This works as follows. For the left- and right-moving Hamiltonians we have the decomposition

$$L_0 = \frac{1}{2}p_L^2 + N, \quad \overline{L}_0 = \frac{1}{2}p_R^2 + \overline{N}, \tag{2.24}$$

where N and $\overline{N}$ denote the remaining left- and right-moving oscillator modes (including those of the $2N_f$ left-moving fermions), and

$$(p_L, p_R) = \frac{1}{\sqrt{\tilde{T}}}\left(p\,\frac{\phi}{2} - w\,\frac{\tilde{T}}{\phi},\, p\,\frac{\phi}{2} + w\,\frac{\tilde{T}}{\phi}\right) \ , \tag{2.25}$$

are winding and momentum excitations along the target space S^1 satisfying the usual relation

$$p_R^2 - p_L^2 = 2\,p\,w \ . \tag{2.26}$$

The $(0, 4)$ elliptic genus projects on the right-moving sector to states which preserve four fermionic operators of the large $\mathcal{N} = 4$ algebra [3, 41, 42]:

$$\overline{L}_0 - c_R/24 = \frac{\left(p\,\frac{\phi}{2} + w\,\frac{\tilde{T}}{\phi}\right)^2}{2\tilde{T}} \ . \tag{2.27}$$

We continue now with comparing these quantum numbers with those of the compactified gauge theory on $\mathbb{R}^3 \times T^2$. For simplicity we let the torus be the direct product $S^1 \times S^1$, where the first S^1 is the Euclidean time circle of radius R_1 and the second S^1 is the spatial circle wrapped by the magnetic string with radius R_2. Therefore, the volume of the torus is given by $V_{T^2} = R_1 R_2$ and the complex structure modulus by $\tau = i\frac{R_1}{R_2} = i\tau_2$.

The 5-dimensional mass of the magnetic string carrying electric and instanton charge takes the form [43]:

$$M_{\text{5d}} = \left| R_2 \frac{T}{\sqrt{2}} + \frac{N'}{R_2} + i(n_e\phi + n_I Z_I) \right|, \tag{2.28}$$

where N' is the momentum around the spatial S^1. We consider the limit $R_2\, T \gg \phi, Z_I, \frac{1}{R_2}$. In this limit, the mass has the following expansion[6]

$$M_{\text{5d}} = R_2 \frac{T}{\sqrt{2}} + \frac{(n_e\phi + n_I Z_I)^2}{\sqrt{2}\, R_2\, T} + \frac{N'}{R_2} + \ldots, \tag{2.29}$$

where the dots represent terms with increasing negative powers in T, which vanish in the limit. The energy of the excitations of the CFT provide the corrections to the leading term $R_2\, T/\sqrt{2}$. This gives for the dimensionless action $R_1\, M_{\text{5d}}$ in terms of the CFT operators:

$$R_1\, M_{\text{5d}} = V_{T_2} \frac{T}{\sqrt{2}} + \tau_2 \left(L_0 - c_L/24 + \overline{L}_0 - c_R/24 \right). \tag{2.30}$$

Using Eqs. (2.4), (2.25) and (2.27) one finds for $r = 1$ the following identification of CFT and 5d parameters:

$$w = n_I\ , \qquad p = n_e + \kappa\, n_I/4\ , \qquad N = N' + pw + c_L/24\ , \qquad \tilde{T} = \frac{T}{2\sqrt{2}}\ , \tag{2.31}$$

that is, the winding number of the magnetic string corresponds to instanton charge and the momentum corresponds to electric charge shifted by a half-integral multiple of instanton charge.

In conclusion, we have derived that the 2d $(0,4)$ CFT for $r = 1$ and $N_f \leq 8$ flavors is an orbifolded free CFT. The winding and momentum charges in this CFT are related to the instanton and electric charges according to (2.31). These facts will be important when we give a proof of our conjecture in Section 4.

2.2 Massive hypermultiplets

There are some important modifications in the analysis of the previous subsection that occur when the hypermultiplets are taken to be massive. We now discuss some of these effects.

In the three-dimensional gauge theory, after compactification of the 5d theory on a torus T^2, we have $2N_f$ (two-component) Dirac species in the fundamental representation, which we denote by $\chi^i, \tilde{\chi}^i; i = 1, ..., N_f$. In the five dimensional theory, they just become N_f Dirac fermions or, equivalently, $2N_f$ symplectic Majorana fermions on which a global flavor symmetry group $SO(2N_f)$ acts.

[6] A very similar expansion for the mass of D4/M5-branes was obtained in [44].

In three dimensions, one can add a triplet of mass terms for each species. First there is a complex mass that can be interpreted as coming from a superpotential in four dimensions. The complex mass terms for the fermions are of the form $M_i \chi^i \tilde{\chi}^i + c.c.$, with M_i complex. Secondly, there is a real mass term, a Dirac mass, which looks like

$$\mathcal{L}_{mass} = -m_i(\bar{\chi}^i \chi^i + \bar{\tilde{\chi}}^i \tilde{\chi}^i) \ . \tag{2.32}$$

Similar mass terms can be written for the hypermultiplet scalars in such a way that the theory remains supersymmetric. In fact, there is a simple way as how the real mass terms arise from a five dimensional point of view. Before adding masses, one can gauge a subgroup of the flavor group $U(1)^{N_f} \subset SO(2N_f)$. To do this, one introduces momentarily N_f five-dimensional vector multiplets, each containing a real scalar field. One then freezes these vector multiplets, by putting them in a background in which the real scalars are non-zero and constant, with a vacuum expectation value equal to the masses m_i. On top of this, we can allow for non-trivial Wilson lines for the flavor gauge fields, after compactification on the torus with complex structure τ. Hence we define

$$y_i \equiv \mathrm{Re}(y_i) + \tau\,\mathrm{Im}(y_i) = 2\oint_{S^1_3} A_i + 2\tau \oint_{S^1_4} A_i \ , \tag{2.33}$$

in the same notation as equation (2.18) of [22]. This procedure preserves supersymmetry and generates mass terms as in (2.32).

Since only real mass terms have a five-dimensional interpretation, we set possible complex masses in three dimensions to zero. Combined with the relevant Yukawa terms for the hypermultiplet fermions, we get terms in the three-dimensional action of the form

$$\mathcal{L}_f = \bar{\chi}^i(\sigma - m_i)\chi^i + \bar{\tilde{\chi}}^i(\sigma - m_i)\tilde{\chi}^i \ . \tag{2.34}$$

Here, σ is the adjoint scalar field, whose vacuum expectation value we have denoted by

$$\sigma = \phi\,\tau_3 \ , \tag{2.35}$$

and without loss of generality we can take $\phi > 0$, as before.

The structure of the zero modes for the hypermultiplet fermions now changes. It follows from index theorems [30] that, for a given value of the flavor index i (see also [45]):

$$\begin{aligned} \phi > |m_i| \qquad &\rightarrow \qquad 2r \text{ fermionic zero modes} \ , \\ \phi < |m_i| \qquad &\rightarrow \qquad \text{no fermionic zero modes} \ , \end{aligned} \tag{2.36}$$

and when $\phi = m_i$ for a particular value of i, the zero mode is still present but becomes non-normalizable, i.e. it is not square-integrable over $\mathbb{R}^3$. So, only when $\phi > |m_i|$ for all values of i, we obtain the same number of fermionic collective coordinates as in (2.8).

One can include the effect of the mass terms on the $(0,4)$ sigma model. In fact, this was done in [28]. However, we will not study the corresponding CFT in this paper and leave this as a topic for future research. The only relevant piece of information for the present discussion, is the jump in the number of fermionic zero modes given by (2.36). As we discuss in the next section, this jump can be interpreted in terms of geometric transitions when embedding the theory in M-theory, indicating that the 5d/2d/4d correspondence also works for massive hypermultiplets.

3 M-theory embedding and $\mathcal{N}=4, d=4$ SYM on $\mathbb{B}_n$

The five-dimensional gauge theory of Section 2 can be embedded into M-theory using either a brane construction [46] or in terms of geometric engineering [20, 21, 47]. For the 4d/2d correspondence we are proposing, the latter construction is more important and we shall follow it in this section. The strategy here will be to identify the magnetic string with the M5-brane wrapped around a del Pezzo surface and then look at two different limits of the M5-brane world-volume theory. To obtain a $\mathcal{N}=1$ gauge theory on $\mathbb{R}^3 \times T^2$ we compactify M-theory on $\mathrm{CY}_3 \times T^2$ where CY_3 is a local Calabi-Yau threefold of the following form

$$\begin{array}{ccc} \mathcal{O}(K) & \longrightarrow & \mathrm{CY}_3 \\ & & \downarrow \\ & & \mathbb{B}_{N_f} \end{array} \tag{3.1}$$

In the above $\mathcal{O}(K)$ is the canonical bundle over the surface $\mathbb{B}_{N_f}$ which itself is the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ blown-up at N_f points. As these surfaces have positive anti-canonical class $-K_{\mathbb{B}_{N_f}} > 0$, they are rigid inside the Calabi-Yau.

3.1 The surface $\mathbb{B}_n$ and lattice $H_2(\mathbb{B}_n, \mathbb{Z})$

In this subsection, we explain in more detail aspects of the lattice $H_2(\mathbb{B}_n, \mathbb{Z})$ and interpret them in terms of the five dimensional field theory. Five dimensional $\mathcal{N}=1$ $SU(2)$ gauge theory is parametrized on the Coulomb branch by the bare coupling constant g_5, the vev ϕ of the vector multiplet scalar, and the bare masses m_i, $i = 1, \cdots, N_f$. All these parameters are related to the geometry of the del Pezzo surface as discussed below.

The homology group of $\mathbb{B}_n$, $H_2(\mathbb{B}_n, \mathbb{Z})$, is generated by the classes $\mathbf{f}$ and $\mathbf{C}$ corresponding to the two $\mathbb{P}^1$'s of $\mathbb{F}_0$ (the first being the fiber $\mathbb{P}^1_f$ and the other the base denoted by $\mathbb{P}^1_B$) and n blow-up classes $\mathbf{c}_i$, $i = 1, \cdots, n$. As $\mathbb{P}^1_f$ and $\mathbb{P}^1_B$ have zero self-intersection and meet each other exactly at one point transversaly, we have

$$\mathbf{C}^2 = \mathbf{f}^2 = 0, \quad \mathbf{C} \cdot \mathbf{f} = 1 \ . \tag{3.2}$$

Furthermore, all other classes have self-intersection number -1 and are mutually orthogonal to each other and to the two classes $\mathbf{C}$ and $\mathbf{f}$. Thus we obtain the following intersection matrix

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & -1 & & \\ & & & \cdots & \\ & & & & -1 \end{pmatrix} . \tag{3.3}$$

In this basis the canonical class of the del Pezzo is given by

$$-K_{\mathbb{B}_n} = 2\mathbf{C} + 2\mathbf{f} - \sum_{i=1}^{n} \mathbf{c}_i \ . \tag{3.4}$$

Geometrically, the del Pezzo surface $\mathbb{B}_n$ can be viewed as a fiber space of $\mathbb{P}^1_B$ where the generic fiber is $\mathbb{P}^1_f$. The sizes of base and fiber $\mathbb{P}^1$ are related to the gauge theory parameters as follows [22]

$$J \cdot \mathbf{f} = 2\phi \ , \quad J \cdot \mathbf{C} = \frac{1}{4g_5^2} + 2\phi \ , \tag{3.5}$$

where J is the pullback of the Calabi-Yau Kähler form to the del Pezzo. In the weak coupling limit the volume of the base becomes proportional to $1/g_5^2$. Moreover, over n points the fiber has two $\mathbb{P}^1$'s intersecting at a point, i.e. the geometry of a resolved A_2 singularity. Denoting each of the blow-up modes for the $\mathbb{P}^1$'s by $[A_2^{i,1}]$ and $[A_2^{i,2}]$ one furthermore arrives at the relation

$$\mathbf{f} = [A_2^{i,1}] + [A_2^{i,2}] \ . \tag{3.6}$$

This relation of the two classes with $\mathbf{f}$ and the conditions

$$[A_2^{i,1}] \cdot [A_2^{i,2}] = 1, \quad [A_2^{i,1}]^2 = [A_2^{i,2}]^2 = -1 \ , \tag{3.7}$$

uniquely fixes these as follows

$$[A_2^{i,1}] = \mathbf{f} - \mathbf{c}_i, \quad [A_2^{i,2}] = \mathbf{c}_i \ . \tag{3.8}$$

The volumes of these classes are related to the $SU(2)$ gauge theory parameters as follows [20, 47]

$$J \cdot [A_2^{i,1}] = \phi + m_i \ , \quad J \cdot [A_2^{i,2}] = \phi - m_i \ , \tag{3.9}$$

where m_i, $i = 1, \ldots, n = N_f$ are hypermultiplet masses. Thus the condition for J to lie in the Kähler cone is:

$$\begin{aligned} 0 &< 2\phi \ , \\ 0 &< \phi + m_i \ , \\ 0 &< \phi - m_i \ . \end{aligned} \tag{3.10}$$

The curves $[A_2^{i,1}]$ or $[A_2^{i,2}]$ can be flopped to give rise to other birational models of the Calabi-Yau X. More precisely, when $\phi = m_i$ one of the $\mathbb{P}^1$'s $[A_2^{i,2}]$ has shrunk to zero size. At this point in moduli space we can do a flop in the Calabi-Yau eliminating this $\mathbb{P}^1$ in the del Pezzo and growing another $\mathbb{P}^1$ in the Calabi-Yau orthogonal to the del Pezzo. In this process the del Pezzo $\mathbb{B}_n$ gets replaced by $\mathbb{B}_{n-1}$. Notice that this is in complete agreement with the 2d side, as one of the flavor fermionic zero modes becomes non-renormalizable and hence disappears from the CFT.

The exceptional divisors of the blow-up maps $\phi_i : \mathbb{B}_i \to \mathbb{B}_{i-1}$ together with the generic fiber class $\mathbf{f}$ and the base class $\mathbf{C}$ span as lattice elements $(\mathbf{C}, \mathbf{f}, \{\mathbf{c}_i\})$ the unimodular lattice $\Lambda_{\mathbb{B}_n} \cong H^2(\mathbb{B}_n, \mathbb{Z})$ whose quadratic form is given by (3.3). In the following, we will abbreviate $\Lambda_{\mathbb{B}_n}$ to Λ. The signature of Λ is thus $(1, b_2 - 1) = (1, n+1)$. We let $G(\Lambda)$ be the Grassmannian of positive definite subspaces of $\Lambda \otimes \mathbb{R}$, this space is $b_2 - 1$ dimensional. A choice of $J \in \Lambda \otimes \mathbb{R}$ such that $J^2 > 0$ corresponds to a point in $G(\Lambda)$, and determines a

split of Λ into positive and negative definite lattices denoted by $\Lambda_\pm$. The projection of a vector $\mathbf{k} \in \Lambda$ to Λ_+ is given by

$$\mathbf{k}_+ = \frac{\mathbf{k} \cdot J}{J^2} J \,, \tag{3.11}$$

and to Λ_- by $\mathbf{k}_- = \mathbf{k} - \mathbf{k}_+$. In this paper we will be mainly dealing with the situation where all mass-parameters m_i, $i = 1, \cdots, N_f$ are zero. In this case the 5d gauge theory exhibits an enhanced $SO(2N_f)$ global flavor symmetry. As was already noted in section 2 this symmetry also shows up in the left-moving fermionic sector of the magnetig string. In order to make this symmetry more manifest we will perform a basis change of the lattice Λ. We want in particular to extract a D_n lattice. To this end we write Λ as a direct sum: $A \oplus D$ with the lattice A spanned by $\mathbf{a}_1 = -K_{\mathbb{B}_n}$ and $\mathbf{a}_2 = \mathbf{f}$, and the lattice D spanned by the remaining n directions denoted by $\boldsymbol{d}_i$. One obtains for the quadratic form in the new basis:

$$\begin{pmatrix} 8-n & 2 & \\ 2 & 0 & \\ & & -\mathcal{Q}_{D_n} \end{pmatrix} , \tag{3.12}$$

where $\mathcal{Q}_{D_n}$ is the D_n Cartan matrix. We will denote the projection of a vector $\mathbf{k} \in \Lambda$ to the lattice A (respectively D) by $\mathbf{k}_A$, respectively $\mathbf{k}_D$.

Thus the uni-modular lattice Λ can be represented as a gluing of the two lattices A and D. Lattice points of Λ can be written as $\boldsymbol{a}+\boldsymbol{d}$, with in general $\boldsymbol{a} \in A^*$ and $\boldsymbol{d} \in D^*$. We are interested in representatives for the coset A^*/A and D^*/D, which are called “glue vectors” of A respectively D [48]. Obtaining the uni-modular lattice Λ from A and D amounts to a choice of isomorphism between A^*/A and D^*/D. Since $\det A = \det D = 4$ this isomorphism is given by four vectors $\boldsymbol{g}_i = (\boldsymbol{a}_i, \boldsymbol{d}_i)$, which we will also refer to as gluing vectors. The glue vectors of A can be found by acting with the inverse $\mathcal{Q}_A^{-1} = \frac{1}{4}\begin{pmatrix} 0 & 2 \\ 2 & n-8 \end{pmatrix}$ on $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, and similarly for those of D.[7] These gluing vectors correspond to correlation of charges in the spectra of the field theories, for example between electric and flavor charge [33, 34], as will be explained in more detail in Section 4.3.

Most qualitative aspects can be understood by taking $n = 1$. The lattice D is generated by $\mathbf{d}_1 = -\mathbf{f} + 2\mathbf{c}_1$. For $A \oplus D$, the gluing vectors are:

$$\begin{aligned} \boldsymbol{g}_0 &= \mathbf{0}, & \boldsymbol{g}_1 &= \tfrac{1}{4}(2,-5,3), \\ \boldsymbol{g}_2 &= \tfrac{1}{4}(0,2,2), & \boldsymbol{g}_3 &= \tfrac{1}{4}(2,-7,1). \end{aligned} \tag{3.13}$$

We observe in particular that $\exp\left(2\pi i \sum_{j=1}^3 \boldsymbol{g}_{i,j}\right) = 1$, and also that $\exp\left(4\pi i \sum_{j=2}^3 \boldsymbol{g}_{i,j}\right) = 1$. We choose as generators for D when $n = 2$:

$$\mathbf{d}_1 = -\mathbf{f} + \mathbf{c}_1 + \mathbf{c}_2, \qquad \mathbf{d}_2 = -\mathbf{f} + 2\mathbf{c}_2, \tag{3.14}$$

[7] The case $n = 0$ needs to be treated separately; since the determinant of the quadratic form is 4 in the new basis, only one vector in A^*/A needs to be added to recover the original lattice. Equivalently one can choose as basis vectors $\mathbf{a}_1 = -\frac{1}{2}K_{\mathbb{B}_0}$ and $\mathbf{a}_2 = \mathbf{f}$.

with gluing vectors:

$$\begin{aligned} \boldsymbol{g}_0 &= \mathbf{0}, & \boldsymbol{g}_1 &= \tfrac{1}{2}(1,1,1,1), \\ \boldsymbol{g}_2 &= \tfrac{1}{2}(0,1,0,1), & \boldsymbol{g}_3 &= \tfrac{1}{2}(1,0,1,0). \end{aligned} \tag{3.15}$$

Similarly for $n = 3$ we have:

$$\mathbf{d}_1 = -\mathbf{f} + \mathbf{c}_1 + \mathbf{c}_2, \qquad \mathbf{d}_2 = -\mathbf{f} + \mathbf{c}_2 + \mathbf{c}_3, \qquad \mathbf{d}_3 = -\mathbf{f} + \mathbf{c}_1 + \mathbf{c}_3, \tag{3.16}$$

and

$$\begin{aligned} \boldsymbol{g}_0 &= \mathbf{0}, & \boldsymbol{g}_1 &= \tfrac{1}{4}(2,-3,1,1,-3), \\ \boldsymbol{g}_2 &= \tfrac{1}{4}(0,2,2,-2,-2), & \boldsymbol{g}_3 &= \tfrac{1}{4}(2,-5,-1,-1,-1). \end{aligned} \tag{3.17}$$

The determination of the gluing vectors finishes our discussion of the basis change (3.12). A direct advantage of the new basis is that we can now easily write down the pullback of the Calabi-Yau Kähler-form to the del Pezzo surface. In the following we will set $n = N_f$ whenever refering to the del Pezzo surface $\mathbb{B}_n$. Expressing the Kähler modulus J in terms of the parameters of the five-dimensional gauge theory one obtains:

$$J = \frac{\mathbf{f}}{4\, g_5^2} - \phi\, K_{\mathbb{B}_n} - \sum_{i,j=1}^{N_f} m_i\, C^{-1}_{D_n,ij}\, \boldsymbol{d}_j. \tag{3.18}$$

where $C^{-1}_{D_n}$ is the inverse of the Cartan matrix C_{D_n}. This is consistent with the hypermultiplet masses as prescribed (3.9).

Finally, we briefly mention a represention of the lattice Λ as a gluing of the 1-dimensional lattice spanned by $K_{\mathbb{B}_n}$ and the E_{n+1}-lattice, where the quadratic form of the E_{n+1}-lattice is given by the Cartan matrices of $E_1 = SU(2)$, $E_2 = SU(2) \times U(1)$, $E_3 = SU(3) \times SU(2)$, $E_4 = SU(5)$, $E_5 = Spin(10)$, and for $n = 6, 7, 8$ E_n are the usual exceptional groups. Bases for E_n are given by:

$$\begin{aligned} &\mathbf{e}_1 = \mathbf{C} - \mathbf{f}, \qquad n = 0, \\ &\left.\begin{aligned} \mathbf{e}_1 &= \mathbf{C} - \mathbf{f}, \\ \mathbf{e}_2 &= \mathbf{c}_1 \end{aligned}\right\} \qquad n = 1, \\ &\left.\begin{aligned} \mathbf{e}_1 &= \mathbf{C} - \mathbf{f} \\ \mathbf{e}_2 &= \mathbf{f} - \mathbf{c}_1 - \mathbf{c}_2 \\ \mathbf{e}_i &= \mathbf{c}_{i-2} - \mathbf{c}_{i-1}, \quad i = 3, \cdots, n+1 \end{aligned}\right\} \qquad n \geq 2 \end{aligned} \tag{3.19}$$

This basis becomes important in the infinite coupling limit $g_5 \to \infty$ and captures the exceptional symmetry of the non-trivial fixed points these theories [19, 20].

3.2 Magnetic strings from M5/$\mathbb{B}_n$

Having identified the parameters of the field theory in terms of moduli of the del Pezzo we now turn to the indentification of states. Here we note from [22] that the W-bosons

correspond to M2-branes wrapped around the $\mathbb{P}^1_f$ and states with topological instanton charge correspond to wrapping around $\mathbb{P}^1_B$. Furthermore, the M5-brane wrapped around the whole del Pezzo corresponds in the 5d gauge theory to the magnetic string with tension

$$\frac{T}{\sqrt{2}} = Z_m = \mathrm{Vol}(\mathbb{B}_{N_f}) = \frac{1}{2}\int_{\mathrm{CY}_3} [\mathbb{B}_{N_f}] \wedge J \wedge J \ , \tag{3.20}$$

as was also shown in [22]. Apart from magnetic charge $n_m = r$, which corresponds to five-brane wrappings, a generic state can have electric n_e, instanton n_I and flavor charge $n_{f,i}$, and is parametrized by an element of the second homology lattice as follows

$$\frac{1}{2} n_e \mathbf{f} - n_I K_{\mathbb{B}_{N_f}} - \frac{1}{2}\sum_{i=1}^{N_f} n_{f,i}\, \boldsymbol{d}_i \in H^2(B_{N_f}, \mathbb{Z}) + r K_{\mathbb{B}_{N_f}}/2 \ . \tag{3.21}$$

The last term is due to the non-integral flux-quantization on the M5-brane [1]. The new ingredients here are the fundamental matter particles which can be identified with M2-branes wrapping the curves $A_2^{i,1}$ as well as $A_2^{i,2}$ with a different orientation. Thus, to summarize, we arrive at Table 1 which gives us the dictionary between field theory and M-theory.

	field theory	M-theory
coupling	$\frac{1}{g^2_{5,0}}$	ϕ_B
moduli	$\phi,\ m_i$	$\phi_f,\ m_i$
states	W-bosons	$M2/\mathbb{P}^1_f$
	fundamental matter	$M2/\{A_2^{i,1} - A_2^{i,2}\}$
	dyonic instantons	$M2/\{n\mathbb{P}^1_B + m\mathbb{P}^1_f\}$
	magnetic string	$M5/\mathbb{B}_{N_f}$

Table 1. Dictionary between five-dimensional field theory and M-theory. In the above n and m are arbitrary integers and refer to the number of wrappings around $\mathbb{P}^1_B$ and $\mathbb{P}^1_f$.

We now claim that in the regime where the characteristic lengths ℓ_{T^2} and $\ell_{\mathbb{B}_{N_f}}$ of T^2 and $\mathbb{B}_{N_f}$ are such that:

$$\ell_{T^2} \gg \ell_{\mathbb{B}_{N_f}}, \tag{3.22}$$

the dynamics of r M5-branes wrapped around $\mathbb{B}_{N_f}$ is governed by the $(0,4)$ sigma model on T^2, whose target space we propose to be equal to that of r $SU(2)$ monopoles introduced in Section 2. The chirality of the sigma model can be derived from the chirality of the $(0,2)$ theory of r M5-branes. Furthermore, the number of supercharges can be deduced consistently from two viewpoints. First of all, as the M5-brane is the solitonic string in 5d $\mathcal{N} = 1$ SYM, its worldvolume preserves half of the supersymmetry giving us 4 supercharges. On the other hand, the topological twist on the del Pezzo surface preserves exactly 4 supercharges due to an argument in [7].

3.3 $\mathcal{N}=4, d=4$ SYM on $\mathbb{B}_n$ from M5/T^2

Compactifying the worldvolume theory of r coincident M5-branes on $T^2 \times \mathbb{B}_{N_f}$ in the regime

$$\ell_{T^2} \ll \ell_{\mathbb{B}_{N_f}} \ , \tag{3.23}$$

will give rise to topologically twisted $\mathcal{N}=4$ $U(r)$ Yang-Mills theory on $\mathbb{B}_{N_f}$ [8] studied in detail in Ref. [7]. In this setup the complex structure τ of T^2 gets identified with the complexified $U(r)$ gauge coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \ . \tag{3.24}$$

The $SL(2,\mathbb{Z})$ symmetry of the T^2 then descends to S-duality acting on the gauge coupling, implying that the partition function of the topological theory transforms as a modular form. In fact, the partition function of this theory localizes on the solutions of least action, known as the BPS solutions, given the topological properties of the fields. These solutions include the instanton (or anti-self dual) solutions: $F = - * F$. Moreover, the coefficients of the partition function can be shown to equal the Euler numbers of instanton moduli spaces if the spaces are smooth [7]. If the moduli space is not smooth, the definition of the suitable Euler number is rather intricate, although it is clear physically that a well-defined integer should exist. We simply refer to this quantity as the “BPS invariant”.

The localization to BPS solutions allows to explicitly compute the partition function using mathematical techniques. Instanton moduli spaces are more abstractly described as moduli spaces of semi-stable coherent sheaves with respect to the Kähler modulus (or polarization) J of the surface, see for example [49]. The Chern character of the sheaf E is determined in terms of the $U(r)$ field strength F by:

$$\mathrm{ch}(E) = r + \frac{i}{2\pi}\mathrm{Tr}\, F + \frac{1}{8\pi^2}\mathrm{Tr}\, F \wedge F.$$

We abbreviate the topological classes of the sheaf by $\Gamma = (r, \mathrm{ch}_1, \mathrm{ch}_2)$. Another often used quantity in the context of sheaves is the discriminant $\Delta(E)$ defined by $\frac{1}{r(E)}(c_2(E) - \frac{r(E)-1}{2r(E)}c_1(E)^2)$. We refer to the references for more details and definitions [11–13].

In this mathematical context it is actually possible to compute instead of the Euler number the more refined Poincaré polynomial, enumerating the Betti numbers of the moduli spaces. This quantity can probably be obtained from the $\mathcal{N}=4$ Yang-Mills path integral by introducing a potential for an R-charge of the theory. We continue by defining the refined BPS invariants $\Omega(\Gamma, w; J)$ in an (informal) way, let $p(\mathcal{M}_J(\Gamma), w) = \sum_{i=0}^{2\dim_{\mathbb{C}}(\mathcal{M}_J(\Gamma))} b_i(\mathcal{M}_J(\Gamma))\, w^i$, be the Poincaré polynomial of the moduli space of semi-stable sheaves for polarization J. Then we define

$$\Omega(\Gamma, w; J) := \frac{w^{-\dim_{\mathbb{C}} \mathcal{M}_J(\Gamma)}}{w - w^{-1}}\, p(\mathcal{M}_J(\Gamma), w). \tag{3.25}$$

The numerical invariant $\Omega(\Gamma; J)$ is defined by the limit $w \to -1$:

$$\Omega(\Gamma; J) = \lim_{w\to -1} (w - w^{-1})\, \Omega(\Gamma, w; J) = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_J(\Gamma)} \chi(\mathcal{M}_J(\Gamma)). \tag{3.26}$$

The rational refined invariants are defined by [11]:

$$\bar{\Omega}(\Gamma,w;J)=\sum_{m|\Gamma}\frac{\Omega(\Gamma/m,-(-w)^m;J)}{m}. \tag{3.27}$$

The generating function of refined BPS invariants for rank r sheaves on the surface S takes the following form [6, 7, 13]:

$$\begin{aligned}\mathcal{Z}_r(y,z,\tau;S,J) = &\sum_{c_1,c_2}\bar{\Omega}(\Gamma,w;J)\,(-1)^{rc_1\cdot K_S}\\ &\times q^{r\Delta(\Gamma)-\frac{r\chi(S)}{24}-\frac{1}{2r}(c_1+rK_S/2)^2_-}\bar{q}^{\frac{1}{2r}(c_1+rK_S/2)^2_+}e^{-2\pi i\bar{y}\cdot(c_1+rK_S/2)},\end{aligned}$$

where $y\in H_2(S,\mathbb{C})$, $w=\exp(2\pi iz)$ and $z\in\mathbb{C}$. Furthermore, K_S is the canonical class, and $\chi(S)$ the Euler number of the surface S. Due to the decomposition $U(r)=U(1)\times SU(r)$, this generating function can be decomposed in a set of theta functions $\Theta_{r,\mu}(y,\tau;S)$, summing over $U(1)$-fluxes, and functions $h_{r,\mu}(z,\tau;S,J)$ capturing the $SU(r)$ sector of the gauge theory with magnetic flux given by μ:

$$\mathcal{Z}_r(y,z,\tau;S,J)=\sum_{\mu\in H^2(S,\mathbb{Z}/r\mathbb{Z})}h_{r,\mu}(z,\tau;S,J)\,\overline{\Theta_{r,\mu}(y,\tau;S)}, \tag{3.28}$$

with $\Theta_{r,\mu}(y,\tau;S)$ given by:

$$\Theta_{r,\mu}(y,\tau;S)=\sum_{\mathbf{k}\in H^2(S,r\mathbb{Z})+rK_S/2+\mu}(-1)^{r\mathbf{k}\cdot K_S}q^{\mathbf{k}_+^2/2r}\bar{q}^{-\mathbf{k}_-^2/2r}e^{2\pi iy\cdot\mathbf{k}}, \tag{3.29}$$

and:

$$h_{r,\mu}(z,\tau;S,J)=\sum_{c_2}\bar{\Omega}(\Gamma,w;J)\,q^{r\Delta(\Gamma)-\frac{r\chi(S)}{24}}. \tag{3.30}$$

Mathematically, this decomposition is the consequence of the isomorphism between moduli spaces of sheaves induced by tensoring the sheaf by a line bundle. In the two-dimensional theory, this decomposition is understood as a spectral flow symmetry.

The partition function of the (numerical) BPS invariants is obtained as the limit $\mathcal{Z}_r(y,\tau;S,J)=\lim_{w\to-1}(w-w^{-1})\,\mathcal{Z}_r(y,z,\tau;S,J)$. Physical arguments such as S-duality suggest that $\mathcal{Z}_r(y,\tau;S,J)$ transforms as a multi-variable Jacobi form of weight $(-\frac{3}{2},\frac{1}{2})$ under $SL(2,\mathbb{Z})$. Since $\Theta_{r,\mu}(y,\tau;S)$ is a vector-valued Jacobi form of weight $(\frac{1}{2},\frac{1}{2}(b_2(S)-1))$, the weight of $h_{r,\mu}(\tau;S,J)$ is $-\chi(S)/2$.

An intriguing relation is the blow-up formula which relates for a surface S and its blow-up $\tilde{S}$ the generating functions $h_{r,\mu}(z,\tau;S,J)$ and $h_{r,\mu}(z,\tau;\tilde{S},J)$[12, 15]. Let $\phi:\tilde{S}\to S$ denote the map between the two surfaces and ϕ^* be the pull-back acting on differential forms. Let moreover $\mathbf{c}$ be the additional two-cycle which arises from the blow-up (the exceptional divisor of ϕ). Then the blow-up formula gives an expression for $h_{r,\phi^*c_1-k\mathbf{c}}(z,\tau;\tilde{S},J)$ in terms of $h_{r,c_1}(z,\tau;S,J)$. We will explain this relation for the greatest common divisor $\gcd(r,c_1)=1$. In this case:

$$h_{r,\phi^*c_1-k\mathbf{c}}(z,\tau;\tilde{S},J)=B_{r,k}(z,\tau)\,h_{r,c_1}(z,\tau;S,J)\ , \tag{3.31}$$

with

$$B_{r,k}(z,\tau)=\frac{1}{\eta(\tau)^r}\sum_{\substack{\sum_{i=1}^r a_i=0\\ a_i\in\mathbb{Z}+\frac{k}{r}}} q^{-\sum_{i<j}a_ia_j}\,w^{\sum_{i<j}a_i-a_j}. \tag{3.32}$$

For $\gcd(r,c_1)>1$ the manipulations are more intricate due to strictly semi-stable sheaves [12, 15].

4 Formulation and tests of the conjecture

We start by explicitly formulating the conjecture. Let $\mathcal{M}_r=\mathbb{R}^3\times(S^1\times\widetilde{\mathcal{M}}_r)/\mathbb{Z}^r$ be the $r-$monopole moduli space. For $r=2$, the relative moduli space $\widetilde{\mathcal{M}}_{r=2}$ is the Atiyah-Hitchin manifold. We claim that the elliptic genus of the $(0,4)$ sigma model with target space $\mathcal{M}_r$ is given by partition function of topologically twisted $\mathcal{N}=4$ Yang-Mills on $\mathbb{B}_0\cong\mathbb{P}^1\otimes\mathbb{P}^1$. This statement in fact also appeared in [22]. Here, we elaborate on it and extend it to include vector bundles over $\mathcal{M}_r$, or, in five dimensional language, flavor hypermultiplets.

Five dimensional gauge theory with N_f hypermultiplets in the fundamental representation gives us naturally an $SO(2rN_f)$ bundle over $\mathcal{M}_r$. The corresponding sigma model contains rN_f chiral bosons (or $2rN_f$ chiral fermions) on the left moving side which are sections of an $SO(2rN_f)$ bundle, as explained in Section 2. There is a global $SO(2N_f)$ flavor symmetry group, and the structure is schematically displayed in (4.1):

$$\begin{array}{c} O(r)\times SO(2N_f)\,. \\ \big\downarrow \\ \mathbb{R}^3\times(S^1\times\widetilde{\mathcal{M}}_r)/\mathbb{Z}^r \end{array} \tag{4.1}$$

Conjecture

The elliptic genus of the $(0,4)$ sigma model based on (4.1) is given by the partition function of $\mathcal{N}=4$ Yang-Mills on $\mathbb{B}_n$.

Clearly, the conjecture simplifies a lot for $r=1$. In that case, the partition function can be explicitly computed on both sides, providing a non-trivial test of the $SO(2N_f)$ information of the 2d theory. We will also discuss the conjecture for $r=2$, and the relation with four-dimensional $\mathcal{N}=2$ gauge theory.

4.1 Derivation for $r=1$

Here we will present a proof of the conjecture for $r=1$. We will start by defining the $(0,4)$ elliptic genus and computing it for the case of the magnetic string sigma model with magnetic monopole charge 1, with action given by (2.20). As a next step we will proceed to determine the $\mathcal{N}=4$ SYM partition function on $\mathbb{B}_{N_f}$ and compare the two results.

(0,4) CFT

The elliptic genus of the $(0,4)$ SCFT is defined by:

$$\mathcal{Z}_{\mathrm{CFT}}(\tau,y)=\mathrm{Tr}_{\mathrm{R}}\left[\frac{1}{2}F^2(-1)^F q^{L_0-\frac{c_L}{24}}\bar{q}^{\bar{L}_0-\frac{c_R}{24}}e^{2\pi i y_i J_0^i}\right]\,. \tag{4.2}$$

Here, F is the fermion number of the adjoint fermions and is inserted to soak up the 4 fermionic zero modes of the monopole background. The J_0^i, $i = 1, \cdots, N_f$ are zero modes of currents corresponding to the Cartan subalgebra of $SO(2N_f)$ and the y_i correspond to Wilson lines as discussed in Section 2.2. As usual, we have that $q = \exp 2\pi i\tau$.

In this section we want to compute the index for charge $r = 1$ monopoles in the background of N_f massless flavors. The index receives contributions from the 3 non-compact bosons whose partition function is given by

$$\mathcal{Z}_{\mathbb{R}^3} = \mathrm{Tr}\left[q^{L_0-\frac{3}{24}}\bar{q}^{\bar{L}_0-\frac{3}{24}}\right] = \frac{1}{\tau_2^{3/2}}\frac{1}{|\eta(\tau)|^6}. \tag{4.3}$$

Furthermore, there is the contribution from the compact scalar parametrized by θ and the $2N_f$ left-moving fermions χ^A, $A = 1, \cdots, 2N_f$, which together form the so called Möbius bundle (in the following abbreviated by "MB"). Its partition function is given by

$$\mathcal{Z}_{\mathrm{MB}} = \mathrm{Tr}\left[\bar{q}^{\bar{L}_0-\frac{1}{24}}q^{L_0-\frac{1+N_f}{24}}e^{2\pi i y_i J_0^i}\right] = \frac{\Theta_{\mathrm{MB}}(\tau,y)}{\overline{\eta(\tau)}\eta(\tau)^{N_f+1}}\ . \tag{4.4}$$

A last factor comes from the 4 right-moving adjoint fermions and is of the form

$$\mathrm{Tr}\left[(-1)^F\bar{q}^{\bar{L}_0-\frac{2}{24}}e^{2\pi i z F}\right] = \frac{\overline{\vartheta_1(\tau,z)}^2}{\overline{\eta(\tau)}^2}\ . \tag{4.5}$$

The elliptic genus (4.2) is now computed by multiplying the three contributions, taking twice the derivative to z at $z = 0$. One obtains:

$$\mathcal{Z}_{\mathrm{CFT}}(\tau) = \frac{\Theta_{\mathrm{MB}}(\tau)}{\eta(\tau)^{N_f+4}}\ . \tag{4.6}$$

The main difficulty lies in the computation of $\mathcal{Z}_{\mathrm{MB}}$. As explained in Section 2.1, the constraint (2.23) leads to an orbifolded CFT, where we mod out by the group $G = \{1, g\}$ generated by the identity and $g = (-)^{n_e+F}$. Denoting the moduli space parametrized by the collective coordinates θ and χ by $\mathcal{M}$ we have to compute

$$\mathcal{Z}_{\mathrm{MB}} = \mathcal{Z}_{\mathcal{M}/G} = \frac{1}{|G|}(\mathcal{Z}_{\mathcal{M}}[{}^1_1] + \mathcal{Z}_{\mathcal{M}}[{}^g_1] + \mathcal{Z}_{\mathcal{M}}[{}^1_g] + \mathcal{Z}_{\mathcal{M}}[{}^g_g])\ ,$$

where $\mathcal{Z}_{\mathcal{M}}[{}^x_y]$ corresponds to the partition function of the magnetic string twisted by $x \in G$ in the time direction and by $y \in G$ in the space direction. Thus we obtain

$$\mathcal{Z}_{\mathcal{M}/G} = \frac{1}{2}\mathrm{Tr}_1\left[(1+(-)^{n_e+F})q^{L_0-\frac{c_L}{24}}\bar{q}^{\bar{L}_0-\frac{c_R}{24}}\right] + \frac{1}{2}\mathrm{Tr}_g\left[(1+(-)^{n_e+F})q^{L_0-\frac{c_L}{24}}\bar{q}^{\bar{L}_0-\frac{c_R}{24}}\right]\ , \tag{4.7}$$

where Tr_1 denotes the trace in the untwisted sector and Tr_g is the trace in the g-twisted sector. Denoting by P_c the projection operator to states satisfying the condition c the trace in the untwisted sector becomes

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}_1[\cdots] = {} & \mathrm{Tr}_1\left[\langle\theta'|\otimes\langle\chi'|P_{\mathrm{even}\ n_e}P_{\mathrm{even}\ F}\ q^{L_0-\frac{c_L}{24}}\bar{q}^{\bar{L}_0-\frac{c_R}{24}}|\theta\rangle\otimes|\chi\rangle\right] \\ & +\mathrm{Tr}_1\left[\langle\theta'|\otimes\langle\chi'|P_{\mathrm{odd}\ n_e}P_{\mathrm{odd}\ F}\ q^{L_0-\frac{c_L}{24}}\bar{q}^{\bar{L}_0-\frac{c_R}{24}}|\theta\rangle\otimes|\chi\rangle\right],\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\mathrm{Tr}_1\left[\langle\theta'|\frac{1}{2}(1+(-)^{n_e})q^{L_0-\frac{c_L}{24}}\bar{q}^{\bar{L}_0-\frac{c_R}{24}}|\theta\rangle\right]\cdot\mathrm{Tr}_1\left[\langle\chi'|\frac{1}{2}(1+(-)^{F})q^{L_0-\frac{c_L}{24}}|\chi\rangle\right]\\
&+\mathrm{Tr}_1\left[\langle\theta'|\frac{1}{2}(1-(-)^{n_e})q^{L_0-\frac{c_L}{24}}\bar{q}^{\bar{L}_0-\frac{c_R}{24}}|\theta\rangle\right]\cdot\mathrm{Tr}_1\left[\langle\chi'|\frac{1}{2}(1-(-)^{F})q^{L_0-\frac{c_L}{24}}|\chi\rangle\right].
\end{aligned}$$

The expression for the twisted sector is completely analogous and is obtained by replacing Tr_1 by Tr_g in the above formula. The untwisted sector corresponds to periodic boundary conditions and can therefore be identified with the Ramond sector. We can now evaluate

$$\begin{aligned}
\mathrm{Tr}_R\left[\langle\chi'|\frac{1}{2}(1+(-)^{F})q^{L_0-\frac{c_L}{24}}e^{2\pi i y_i J_0^i}|\chi\rangle\right] &= \frac{1}{2}\left[\prod_{i=1}^{N_f}\frac{\vartheta_2(\tau,y_i)}{\eta(\tau)}+\prod_{i=1}^{N_f}\frac{\vartheta_1(\tau,y_i)}{\eta(\tau)}\right],\\
\mathrm{Tr}_R\left[\langle\chi'|\frac{1}{2}(1-(-)^{F})q^{L_0-\frac{c_L}{24}}e^{2\pi i y_i J_0^i}|\chi\rangle\right] &= \frac{1}{2}\left[\prod_{i=1}^{N_f}\frac{\vartheta_2(\tau,y_i)}{\eta(\tau)}-\prod_{i=1}^{N_f}\frac{\vartheta_1(\tau,y_i)}{\eta(\tau)}\right].
\end{aligned} \tag{4.8}$$

The expressions for the twisted sector are obtained by noting that the mode expansion of the χ^A becomes shifted by half-integers due to

$$\chi^A(x+2\pi,t)=-\chi^A(x,t),\quad A=1,\cdots,2N_f. \tag{4.9}$$

Therefore, the trace has to be computed in the NS sector:

$$\begin{aligned}
\mathrm{Tr}_{NS}\left[\langle\chi'|\frac{1}{2}(1+(-)^{F})q^{L_0-\frac{c_L}{24}}e^{2\pi i y_i J_0^i}|\chi\rangle\right] &= \frac{1}{2}\left[\prod_{i=1}^{N_f}\frac{\vartheta_3(\tau,y_i)}{\eta(\tau)}+\prod_{i=1}^{N_f}\frac{\vartheta_4(\tau,y_i)}{\eta(\tau)}\right],\\
\mathrm{Tr}_{NS}\left[\langle\chi'|\frac{1}{2}(1-(-)^{F})q^{L_0-\frac{c_L}{24}}e^{2\pi i y_i J_0^i}|\chi\rangle\right] &= \frac{1}{2}\left[\prod_{i=1}^{N_f}\frac{\vartheta_3(\tau,y_i)}{\eta(\tau)}-\prod_{i=1}^{N_f}\frac{\vartheta_4(\tau,y_i)}{\eta(\tau)}\right].
\end{aligned} \tag{4.10}$$

Next, we want to compute the trace of the compact boson sector. In order to proceed we recall here some facts about translation orbifold blocks for the compact boson CFT (see for example Appendix B of [50]). This CFT is a 1-dimensional toroidal CFT with symmetry group $O(1,1)$. The transformations associated with it are arbitrary lattice translations which act on a state with momentum p and winding number w as

$$g_{\text{translations}}=\exp\left[2\pi i(p\phi_1+w\phi_2)\right], \tag{4.11}$$

where ϕ_1 and ϕ_2 are rational numbers. This results in a freely acting discrete group of finite order N. When modding out by this symmetry the resulting orbifold has twisted sectors with shifted momentum and winding modes of the form

$$p\mapsto p+n\phi_2,\quad w\mapsto w+n\phi_1,\quad n=0,\cdots,\ N-1. \tag{4.12}$$

In our case we are modding out by the symmetry

$$g = \exp\left[2\pi i \frac{n_e}{2}\right] = \exp\left[2\pi i\left(\frac{p}{2} - \frac{8-N_f}{4}w\right)\right], \tag{4.13}$$

where use has been made of the identities (2.31). This results in the following shifts of p and w in the twisted sector:

$$p \mapsto p + \frac{8-N_f}{4}, \quad w \mapsto w + \frac{1}{2}. \tag{4.14}$$

Now we are in the position to write down the full partition function. It will consist of four pieces, the untwisted sector with even n_e and odd n_e, as well as the twisted sector with even n_e and odd n_e. The difference between the twisted and untwisted sector will be the shift (4.14) in momentum and winding number. The full result then looks as follows

$$\begin{aligned} \mathcal{Z}_{\mathcal{M}/G} = \mathcal{Z}_{S^1}^{\text{untw, } n_e \text{ even}} \cdot \frac{1}{2}\left[\left(\frac{\vartheta_2}{\eta}\right)^{N_f} + \left(\frac{\vartheta_1}{\eta}\right)^{N_f}\right] \\ + \mathcal{Z}_{S^1}^{\text{untw, } n_e \text{ odd}} \cdot \frac{1}{2}\left[\left(\frac{\vartheta_2}{\eta}\right)^{N_f} - \left(\frac{\vartheta_1}{\eta}\right)^{N_f}\right] \\ + \mathcal{Z}_{S^1}^{\text{tw, } n_e \text{ even}} \cdot \frac{1}{2}\left[\left(\frac{\vartheta_3}{\eta}\right)^{N_f} + \left(\frac{\vartheta_4}{\eta}\right)^{N_f}\right] \\ + \mathcal{Z}_{S^1}^{\text{tw, } n_e \text{ odd}} \cdot \frac{1}{2}\left[\left(\frac{\vartheta_3}{\eta}\right)^{N_f} - \left(\frac{\vartheta_4}{\eta}\right)^{N_f}\right] \end{aligned} \tag{4.15}$$

The above can be written more conveniently in terms of $\mathcal{Z}_{S^1}^{a,b}$ where $a, b \in \{0, 1\}$. In this notation $a = 0$ stands for the untwisted sector and $a = 1$ for the twisted sector, $b = 0$ corresponds to even n_e and $b = 1$ to odd n_e. Then we have

$$\mathcal{Z}_{S^1}^{a,b} = \frac{1}{|\eta(\tau)|^2}\left[\sum_{\substack{w \in \mathbb{Z} + \frac{a}{2}\,, \\ p \in \frac{(8-N_f)}{2} w + 2\mathbb{Z} + b + \frac{1}{4}(1-(-)^{N_f})(1-a)}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}\right]. \tag{4.16}$$

The shift $\frac{1}{4}(1-(-)^{N_f})(1-a)$ is included to make sure that for odd N_f the constraint $2pw \in 2\mathbb{Z}$ is satisfied in the untwisted sector.

$\mathcal{N} = 4$ Yang-Mills

We will now determine the partition function $\mathcal{Z}_r(y, z, \tau; \mathbb{B}_n, J)$ (3.28) for $r = 1$. These simplify considerably: the sum over $\Lambda/r\Lambda$ reduces to a single term, and $h_{1,c_1}(z, \tau; \mathbb{B}_n)$ does not depend on J since all rank 1 sheaves are stable. The moduli spaces of these sheaves correspond to the Hilbert scheme of points on $\mathbb{B}_n$. Therefore, $h_{1,c_1}(z, \tau; \mathbb{B}_n)$ is given by [14]:

$$h_{1,c_1}(z, \tau; \mathbb{B}_n) = \frac{i}{\theta_1(\tau, 2z)\, \eta(\tau)^{b_2(\mathbb{B}_n)-1}}.$$

This gives for the numerical invariants:

$$h_{1,c_1}(\tau;\mathbb{B}_n)) = \frac{1}{\eta(\tau)^{\chi(\mathbb{B}_n)}} = \frac{1}{\eta(\tau)^{4+n}} \ , \tag{4.17}$$

since $\chi(\mathbb{B}_n) = 2 + b_2(\mathbb{B}_n) = 4 + n$.

Next, we discuss in more detail the other functions in (3.28), namely the theta functions $\Theta_{r,\mu}(y,\tau;\mathbb{B}_n)$ summing over the lattice Λ. In order to relate these functions to the two-dimensional results, we consider Λ as the gluing of the lattices A and D as described in Section 3.1. In order to compare with the two-dimensional result we choose the parametrization (3.18) for J and set the masses m_i to zero. Then the $\Theta_{r,\mu}(y,\tau;\mathbb{B}_n)$ factorizes as the restriction of J to D vanishes; it takes the form:

$$\Theta_{r,\mu}(y,\tau;\mathbb{B}_n) = \sum_{i=0}^{3} \Theta_{rA,\boldsymbol{g}_i+\mu}(y,\tau)\, \overline{\Theta_{rD_n,\boldsymbol{g}_i+\mu}(y,\tau)}, \tag{4.18}$$

where $\boldsymbol{g}_i$ are the gluing vectors. Moreover, $\Theta_{rA,\mu}(y,\tau)$ is given by:

$$\Theta_{rA,\mu}(y,\tau) = \sum_{\mathbf{k}\in\frac{1}{2}\mathbf{a}_1+\mu+(r\mathbb{Z})^2} (-1)^{r\mathbf{a}_1\cdot\mathbf{k}} q^{\frac{\mathbf{k}_+^2}{2r}}\, \bar{q}^{-\frac{\mathbf{k}_-^2}{2r}}\, e^{2\pi i y\cdot\mathbf{k}} \tag{4.19}$$

where the quadratic form $\mathbf{k}^2$ is obtained from the A matrix (3.12). Similarly, $\Theta_{rD_n,\mu}(y,\tau)$ is given by

$$\Theta_{rD_n,\mu}(y,\tau) = \sum_{\mathbf{k}\in\mu+(r\mathbb{Z})^n} q^{\frac{\mathbf{k}^2}{2r}}\, e^{2\pi i y\cdot\mathbf{k}}, \tag{4.20}$$

and the quadratic form $\mathbf{k}^2$ is here obtained from the D_n Cartan matrix. We have left implicit in the formulas above that $\boldsymbol{g}_i + \mu$ and y should be restricted to the lattice A and D for $\Theta_{rA,\boldsymbol{g}_i+\mu}(y,\tau)$ and $\overline{\Theta_{rD_n,\boldsymbol{g}_i+\mu}(y,\tau)}$ respectively.

For $r = 1$ and $y = 0$, one finds for the D_n theta functions the following:

$$\Theta_{D_n,\boldsymbol{g}_0}(\tau) = \frac{1}{2}\left(\vartheta_3^n + \vartheta_4^n\right) = 1 + 2n(n-1)\,q + \dots, \tag{4.21}$$

$$\Theta_{D_n,\boldsymbol{g}_1}(\tau) = \frac{1}{2}\vartheta_2^n = 2^{n-1}q^{\frac{n}{8}} + \dots, \tag{4.22}$$

$$\Theta_{D_n,\boldsymbol{g}_2}(\tau) = \frac{1}{2}\left(\vartheta_3^n - \vartheta_4^n\right) = 2n\,q^{\frac{1}{2}} + \dots, \tag{4.23}$$

$$\Theta_{D_n,\boldsymbol{g}_3}(\tau) = \frac{1}{2}\vartheta_2^n = 2^{n-1}q^{\frac{n}{8}} + \dots, \tag{4.24}$$

i.e. precisely the theta functions of the current algebra $\widehat{so}(2n)_1$. The first coefficients in the q-expansion are the dimensions of $SO(2n)$ representations. Comparing with (4.10), we observe that these D_n theta functions precisely correspond to the ones obtained from the various orbifold sectors in two dimensions!

It remains, for the identification of the four-dimensional with the two-dimensional partition function, to identify the $\Theta_{A,\boldsymbol{g}_i}$ with the different twisted and untwisted theta-functions of $\mathcal{Z}_{S^1}$. In order to perform this comparison we need to rewrite the summation

in $\Theta_{A,\boldsymbol{g}_i}$ in the same units as the summation involved in $\mathcal{Z}_{S^1}$. This means identifying the vector $\mathbf{k} \in H^2(\mathbb{B}_n, \mathbb{Z}) + \frac{rK_{\mathbb{B}_n}}{2}$ with the charge vector of the five-dimensional field theory as in (3.21).

In this subsection, we will work in the massless case, $m_i = 0$ and for magnetic charge $r = 1$. One then easily computes with (3.18)

$$\frac{1}{2}J^2 = \frac{T}{\sqrt{2}} \ , \qquad J \cdot \mathbf{k} = n_I Z_I + n_e \phi \ , \qquad -\mathbf{k} \cdot K_{\mathbb{B}_{N_f}} = n_e + (8 - N_f) n_I \ , \tag{4.25}$$

so we recover in the second equation the central charge of the dyonic instanton with instanton charge n_I and electric charge n_e. It now follows straightforwardly that

$$\mathbf{k}_+^2 = \frac{(n_e \phi + n_I Z_I)^2}{\sqrt{2} T} \ , \tag{4.26}$$

and furthermore, we have the important identity

$$\mathbf{k}_{|A}^2 = \mathbf{k}_+^2 - (-\mathbf{k}_{|A,-}^2) = 2n_I \Big(n_e + \frac{(8 - N_f)}{2} n_I \Big) \ , \tag{4.27}$$

where we have restricted the charge vector $\mathbf{k}$ to the lattice Λ_A, which is needed in the calculation of the partition function. Comparing the partition function on the A-lattice (4.19) with the partition function for a conformal field theory, we identify left- and right moving momenta as

$$p_R^2 = \mathbf{k}_+^2 \ , \qquad p_L^2 = -\mathbf{k}_{|A,-}^2 \ , \tag{4.28}$$

with the identification of p_R and p_L as in (2.25).

To compute the partition function more explicitly, we have to find the theta-functions corresponding to the different gluing vectors $\boldsymbol{g}_i$ restricted to the A-lattice and identify them with twisted and untwisted sectors. For the vector $\mathbf{k}$ restricted to the A-lattice $\mathbf{k}_A$, this implies that the coefficients are rational:

$$\mathbf{k}_A \in (\tfrac{1}{2}, 0) + \boldsymbol{g}_{i,A} + A \ , \tag{4.29}$$

where $\boldsymbol{g}_i; i = 0, ..., 3$ is one of the four gluing vectors. Comparing with (3.21) gives for the charges

$$n_I \in \frac{1}{2} + \boldsymbol{g}_{i,I} + \mathbb{Z} \ , \qquad n_e \in 2\boldsymbol{g}_{i,e} + 2\mathbb{Z} \ , \tag{4.30}$$

where $\boldsymbol{g}_{i,I}$ is the I-component of the gluing vector $\mathbf{g}_i$, etc. Accordingly, Eq. (4.28) implies that also the winding and momentum modes are rational:

$$w \in \frac{1}{2} + \boldsymbol{g}_{i,I} + \mathbb{Z} \ , \qquad p \in 2\boldsymbol{g}_{i,e} + \frac{(8 - N_f)}{2} \Big(\frac{1}{2} + \boldsymbol{g}_{i,I} \Big) + \mathbb{Z} \ . \tag{4.31}$$

We now look in more detail at the properties of the gluing vectors that are given in Section 3.1. We split them into two sectors, the twisted sector, defined by $\boldsymbol{g}_0$ and $\boldsymbol{g}_2$, and the untwisted sector, spanned by $\boldsymbol{g}_1$ and $\boldsymbol{g}_3$. These sectors will correspond to the twisted and untwisted sectors of the orbifold CFT to which we compare at the end of this subsection.

In the twisted sector, we see that $\boldsymbol{g}_{i,I}$ is always zero, so we get half-integer shifts of the winding modes. Furthermore, in this sector $2\boldsymbol{g}_{0,e} = 0$ and $2\boldsymbol{g}_{2,e} = 1$. Therefore, the twisted sector corresponds to $n_I \in \mathbb{Z} + \frac{1}{2}$ and further splits into two sub-sectors with even and odd n_e. So the shifts we obtain for w and p are

$$\textit{twisted sector}: \qquad w \in \frac{1}{2} + \mathbb{Z}\,, \qquad p \in \pm\frac{N_f}{4} + \mathbb{Z}\,. \tag{4.32}$$

Comparing to the 2d computation we see that this exactly matches the result (4.16)!

In the untwisted sector, we have $\boldsymbol{g}_{i,I} = 1/2$ such that $n_I \in \mathbb{Z}$. For the electric component, one can check explicitly that for $N_f = 0, 1, 2, 3$, the momentum mode gets shifted by an amount that can be absorbed in either an even shift in n_e or an integer shift in n_I. Hence effectively, we can drop these shifts in the lattice sum. We conclude that in the untwisted sector

$$\textit{untwisted sector}: \qquad w \in \mathbb{Z}\,, \qquad p \in \mathbb{Z}, \quad \text{or} \quad p \in \frac{N_f}{2} + \mathbb{Z}\,. \tag{4.33}$$

Again we see that we obtain the same mode expansion as encoded in (4.16). Also, one observes that the difference between $\boldsymbol{g}_{1,e}$ and $\boldsymbol{g}_{3,e}$ just results in a shift of n_e by 1, leading thus to the two subsectors with odd and even n_e. We therefore see that the two partition functions exactly match and thereby confirm our conjecture for $r = 1$.[8]

4.2 Rank $r \geq 2$

This section discusses the proposed correspondence for higher rank $r \geq 2$. On the $\mathcal{N} = 4$ Yang-Mills side, the partition functions for $r \geq 2$ can be determined explicitly. Unfortunately, this is not feasible on the two-dimensional side. Therefore, the partition function of $\mathcal{N} = 4$ Yang-Mills provides a prediction for the elliptic genus of the $(0, 4)$ sigma model.

The partition functions for $r \geq 2$, can be determined using the techniques developed in [11, 12, 15, 16]. The key property of the computation is that $\mathbb{F}_0$ is the product $\mathbb{P}^1 \otimes \mathbb{P}^1$ (and more generally for Hirzebruch surfaces $\mathbb{F}_n$ it is a fibration). The BPS invariants can be determined if one chooses the Kähler modulus J such that one $\mathbb{P}^1$ (the fibre $\mathbf{f}$) is infinitesimally small compared to the other $\mathbb{P}^1$ (the base $\mathbf{C}$). This choice of J is called a suitable polarization. Using the jump of the invariants across walls of marginal stability [15, 51, 52], one can consequently compute the invariants for other choices of J. The BPS-invariants of $\mathbb{B}_n$ are determined from those of $\mathbb{F}_0$ using the blow-up formula (3.31).

We start with the computation of the partition functions for $\mathbb{B}_0$. The polarization J is parametrized by $J_{m_1,m_2} = m_1\mathbf{C} + m_2\mathbf{f}$, such that the suitable polarization is given by $J_{\varepsilon,1}$ with $0 < \varepsilon \ll 1$. For this choice the BPS-invariants vanish for all sheaves with $\gcd(r, \mathbf{f} \cdot c_1) = 1$. If this condition is not satisfied, the BPS-invariants do not vanish and their computation is more involved due to the presence of strictly semi-stable sheaves

[8] As the 2d partition function does not contain a parameter y, we set $y = -\frac{1}{2}\mathbf{a}_1$ to cancel the factor $(-1)^{\mathbf{a}_1\cdot\mathbf{k}}$ in the definition of $\Theta_A(y, \tau)$.

[12, 16]. One finds for $h_{2,c_1}(z,\tau;J_{\varepsilon,1},\mathbb{B}_0)$:

$$\begin{aligned}
&\beta = 1 \mod 2: \\
&\quad h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(z,\tau;J_{\varepsilon,1},\mathbb{B}_0) = 0, \\
&(\alpha,\beta) = (1,0) \mod 2: \\
&\quad h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(z,\tau;J_{\varepsilon,1},\mathbb{B}_0) = \frac{-i\,\eta(\tau)}{\vartheta_1(\tau,2z)^2\vartheta_1(\tau,4z)} + \frac{w^2}{1-w^4}h_{1,0}(z,\tau;\mathbb{B}_0)^2, \\
&(\alpha,\beta) = (0,0) \mod 2: \\
&\quad h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(z,\tau;J_{\varepsilon,1},\mathbb{B}_0) = \frac{-i\,\eta(\tau)}{\vartheta_1(\tau,2z)^2\vartheta_1(\tau,4z)} + \left(\frac{1}{1-w^4}-\frac{1}{2}\right)h_{1,0}(z,\tau;\mathbb{B}_0)^2.
\end{aligned} \tag{4.34}$$

Note that the coefficients of $h_{2,\mathbf{0}}(z,\tau;J_{\varepsilon,1})$ are rational due to the multi-covering formula (3.27). The modular properties of this generating function appear to be more elegant than the ones for integer invariants. However, why these rational coefficients appear in the generating function from the point of view of the $\mathcal{N}=4$ Yang-Mills theory or the two-dimensional field theory is not well understood.

The partition function for more general J_{m_1,m_2} is given by:

$$\begin{aligned}
h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(z,\tau;J_{m_1,m_2},\mathbb{B}_0) &= h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(z,\tau;J_{\varepsilon,1};\mathbb{B}_0) \\
&+\frac{1}{4}h_{1,0}(z,\tau;\mathbb{B}_0)^2 \sum_{(a,b)=-(\alpha,\beta) \mod 2} (\mathrm{sgn}(bm_2-am_1)-\mathrm{sgn}(b-a\varepsilon)) \\
&\times\left(w^{2b-2a}-w^{-2b+2a}\right)q^{\frac{1}{2}ab}.
\end{aligned} \tag{4.35}$$

One can verify that for $J_{1,\varepsilon}$ this function vanishes for $\alpha = 1 \mod 2$, which is expected due to the symmetry of $\mathbf{f}$ and $\mathbf{C}$. Of special interest is the choice $(m_1,m_2)=(1,1)$ which corresponds to $J=-K_{\mathbb{F}_0}$. At this point, a geometric sum can be a carried out in (4.35) giving:

$$\begin{aligned}
h_{2,0}(z,\tau;J_{1,1},\mathbb{B}_0) &= \frac{-i\eta(\tau)}{\vartheta_1(\tau,2z)^2\,\vartheta_1(\tau,4z)} \\
&\quad +h_{1,0}(z,\tau;\mathbb{B}_0)^2\left(\sum_{b\text{ even}}\frac{q^{\frac{1}{2}b^2}}{1-w^4q^b}-\frac{1}{2}q^{\frac{1}{2}b^2}\right) \\
h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(z,\tau;J_{1,1},\mathbb{B}_0) &= h_{1,0}(z,\tau)^2\left(\sum_{b\text{ odd}}\frac{q^{\frac{1}{2}b^2+\frac{1}{2}b}w^2}{1-w^4q^b}\right), \quad \alpha+\beta=1 \\
h_{2,\mathbf{C}-\mathbf{f}}(z,\tau;J_{1,1},\mathbb{B}_0) &= h_{1,0}(z,\tau;\mathbb{B}_0)^2\left(\sum_{b\text{ odd}}\frac{q^{\frac{1}{2}b^2}}{1-w^4q^b}-\frac{1}{2}q^{\frac{1}{2}b^2}\right).
\end{aligned} \tag{4.36}$$

The functions in brackets are (up to multiplication by a theta function) specializations of the Lerch-Appell sum [53]:

$$\mu(u,v;\tau) = \frac{e^{i\pi u}}{\theta_1(v;\tau)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi inv}}{1-e^{2\pi in\tau+2\pi iu}}. \tag{4.37}$$

The surfaces $\mathbb{B}_n$ are obtained by blowing up n points of $\mathbb{F}_0$. Section 3.1 discussed how the lattice $H_2(\mathbb{B}_n, \mathbb{Z})$ is a gluing of a two-dimensional lattice A and a n-dimensional lattice D. Therefore, the polarization J can vary in more directions. We consider here only variations of J in the sublattice A and parametrize J by $J_{m_1,m_2} = m_1\,\mathbf{a}_1 + m_2\,\mathbf{a}_2 \propto -\phi\, K_{B_n} + \frac{1}{4g_5^2}\mathbf{f}$, and the hypermultiplets are thus massless. As for $\mathbb{F}_0$, we start close to a boundary of the Kähler cone, $J_{\varepsilon,1}$, where the BPS-invariants vanish if $\gcd(r, c_1 \cdot \mathbf{a}_2) > 1$. They again do not vanish if this condition is not satisfied, and can then be determined exactly. For illustration, we give two examples:

$$\begin{aligned} h_{2,\mathbf{0}}(z,\tau;J_{\varepsilon,1},\mathbb{B}_n) &= h_{2,c_1}(z,\tau;J_{0,1},\mathbb{B}_n) \\ &\quad + \left(\frac{1}{1-w^4}-\frac{1}{2}\right)\Theta_{2D_n,\mathbf{0}}(\tau) + \frac{w^2}{1-w^4}\,\Theta_{2D_n,\mathbf{1}}(\tau). \end{aligned} \tag{4.38}$$

and for $c_1 = \frac{n}{2}\mathbf{a}_2 - \frac{1}{2}\sum_{i=1}^n \boldsymbol{d}_i$:

$$h_{2,c_1}(z,\tau;J_{\varepsilon,1},\mathbb{B}_n) = h_{2,c_1}(z,\tau;J_{0,1},\mathbb{B}_n) + \frac{w}{1-w^2}\,\Theta_{2D_n,\frac{1}{2}\mathbf{1}}(\tau). \tag{4.39}$$

where

$$h_{2,c_1}(z,\tau;J_{0,1},\mathbb{B}_n) = \frac{-i\,\eta(\tau)}{\theta_1(\tau,2z)^2\,\theta_1(\tau,4z)}\prod_{i=1}^n B_{2,k_i}(z,\tau),$$

with $k_i = c_1 \cdot \mathbf{c}_i$. The product over $i = 1,\dots,n$ on the right hand side is due to the blow-up formula (3.31). Similarly to $n = 0$, the point $J = -K_{B_n}$ is special for $n > 0$,: the sum over walls between $J_{\varepsilon,1}$ and $-K_{B_n}$ can be resummed to a specialization of a Lerch-Appel function multiplied by a theta function.

One can understand for general r the blow-up formula from the two-dimensional perspective from the $so(2rN_f)$ current algebra arising from the $SO(2rN_f)$ bundle over the monopole moduli space. The corresponding theta function sums over an rN_f-dimensional lattice. The sum over an N_f-dimensional sublattice gives $\Theta_{rD_n,\mu}(y,\tau)$, and the remaining $(r-1)N_f$ directions provide the theta functions multiplying $h_{r,c_1}(z,\tau;J_{0,1},\mathbb{B}_0)$.

The parameter w does not appear in the elliptic genus, therefore one should specialize to numerical invariants by taking the limit $w \to -1$. For example for $\mathbb{F}_0$ one finds for $J_{\varepsilon,1}$:

$$\begin{aligned} h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(\tau;J_{\varepsilon,1}) &= 0, && \beta = 1 \mod 2 \\ h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(\tau;J_{\varepsilon,1}) &= -\frac{E_2(\tau)-1}{12\,\eta(\tau)^8}, && (\alpha,\beta) = (1,0) \mod 2, \\ h_{2,\beta\mathbf{C}-\alpha\mathbf{f}}(\tau;J_{\varepsilon,1}) &= -\frac{E_2(\tau)+2}{12\,\eta(\tau)^8}, && (\alpha,\beta) = (0,0) \mod 2, \end{aligned} \tag{4.40}$$

where $E_2(\tau)$ is the Eisenstein series of weight 2. For $J_{1,1}$, the coefficients can be expressed in terms of class numbers $H(n)$ (which count binary quadratic forms with given discriminant [54]):

$$h_{2,\mathbf{C}-\mathbf{f}}(z,\tau;J_{1,1}) = \frac{\vartheta_2(\tau)\sum_{n\geq 0} H(-1+8n)\,q^n}{\eta(\tau)^8}. \tag{4.41}$$

An important aspect of the functions above is that they do not quite transform as a modular form of weight $-\chi(\mathbb{B}_n)/2 = -2 - n/2$ as the partition function for $r = 1$. This is easily seen from Eqs. (4.40) and (4.41), since $E_2(\tau)$ and the class number generating function[9] $\mathfrak{h}(\tau) = \sum_{\substack{n=0 \\ n=0,3 \mod 4}}^{\infty} H(n)q^n$ transform only as a modular form after addition of a suitable non-holomorphic term:

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}, \qquad \hat{\mathfrak{h}}(\tau) = \mathfrak{h}(\tau) + \frac{(1+i)}{8\pi} \int_{-\bar{\tau}}^{i\infty} \frac{\vartheta_3(u)}{(\tau+u)^{3/2}} du. \tag{4.42}$$

Using the techniques of [53], the required non-holomorphic terms can be derived for general values of J and also $w \neq -1$.

Our proposal states that the partition functions above appear as the elliptic genus of a (0,4) sigma model. For $r \geq 2$, however, new issues appear which make the proposal more involved. An important issue is how the dependence of $h_{r,c_1}(\tau; J)$ on J for $r > 1$ is realized on the two-dimensional side. The partition functions for weak coupling or equivalently $J = J_{\varepsilon,1}$ do not have a form which is familiar from conformal field theory. On the other hand, Appell-Lerch functions which appear for $J = -K_{\mathbb{B}_n}$ do appear as characters in conformal field theory [55–57], and interestingly also as partition functions of CFT's with a non-compact field space [58]. Non-holomorphic terms as the integral in (4.42) are argued to be a direct consequence of the (regularization of) non-compact target space. This is in nice agreement with our findings, since the Atiyah-Hitchin moduli space is also non-compact. The structure of the $\mathcal{N} = 4$ Yang-Mills partition function therefore indicates that the elliptic genus of the (0,4) CFT corresponds to the polarization $J = -K_{\mathbb{B}_n}$, and that the two-dimensional field theory dual to Yang-Mills theory for other values of J is not conformal. Around (3.19) is explained that this point $J = -K_{\mathbb{B}_n}$ is also very special in the five-dimensional theory, where it corresponds to E_{n+1} RG fixed point at infinite coupling. Note that the jumps in the quantum spectrum of $\mathcal{N} = 4$ Yang-Mills correspond in the 2d theory to changes of the spectrum along its RG-flow. The Yang-Mills partition function provides an interesting tool for studying such flows.

The above point of view is also consistent with the attractor flow of the moduli in five-dimensional supergravity solutions sourced by M5-branes. In the near horizon AdS_3 region, the moduli are fixed at their attractor point. To see the attractor flow one needs to move out of the near-horizon region. From the point of view of the dual conformal field theory, perturbing the moduli away from the attractor point corresponds to an irrelevant perturbation of the conformal field theory [59]. In our local Calabi-Yau manifold, the attractor point determines J to be proportional to $-K_{\mathbb{B}_n}$. (Recall that for $\mathcal{N} = 4$ Yang-Mills only the direction of J in the Kähler moduli space is relevant.) Thus the supergravity viewpoint suggests that perturbing J away from $-K_{\mathbb{B}_n}$ correspond to perturbing away from the IR fixed point.

In the above, we have concentrated on $r \leq 2$. For $r > 2$ one can also compute explicitly the (holomorphic part of the) partition functions of $\mathcal{N} = 4$ Yang-Mills. The indefinite theta functions are rather involved [11, 13]. It would be very interesting to relate these functions to those of a conformal field theory.

[9]The notation here means to sum over n from 0 to ∞ where n is either 0 or 3 modulo 4.

4.3 Relation with $\mathcal{N} = 2$ Yang-Mills theory in four dimensions

Compactification of the five-dimensional theory on a circle with radius $R_2 \to 0$ gives in four dimensions $\mathcal{N} = 2$, $SU(2)$ gauge theory with N_f hypermultiplets. The monopole string of the five-dimensional theory corresponds to a BPS-monopole or dyon preserving half of the $\mathcal{N} = 2$ supersymmetry of the four dimensional theory. The spectrum of these BPS-monopoles and dyons is fully known. We point out in this section that these spectra are consistent with the partition functions of $\mathcal{N} = 4$ Yang-Mills computed in Subsection 3.3.

The monopole and dyon spectrum of $\mathcal{N} = 2$ gauge theory with N_f flavors can in principle be determined by computing the Dirac index of the monopole moduli space twisted by the connection coming from the flavor fermions [31–33], and is consistent with later analysis [60–62]. We will list here the spectra for $N_f \leq 3$. The W-boson has always charge $(n_m, n_e) = (0, 2)$, and is a singlet of the $SO(2N_f)$ flavor group. For $N_f = 0$, the spectrum consists furthermore of the infinite set of dyons with charge $(1, 2n)$, $n \in \mathbb{Z}$ [63]. The BPS-index Ω (at weak coupling) of the W-boson is equal to -2, and that of the monopole and dyons is equal to 1 [64]. For $N_f = 1$, the spectrum of the quarks and dyons is given by (see e.g. [62]):

Particle (n_m, n_e)	$SO(2)$ charge
Quarks $(0, 1)$	± 1
Dyons $(1, 2n + 1/2)$	$\frac{1}{2}$
$(1, 2n + 3/2)$	$-\frac{1}{2}$

for $N_f = 2$:

Particle (n_m, n_e)	Rep. $SO(4) \cong SU(2) \otimes SU(2)$
Quarks $(0, 1)$	$(\mathbf{2}, \mathbf{2})$
Dyons $(1, 2n)$	$(\mathbf{2}, \mathbf{1})$
$(1, 2n + 1)$	$(\mathbf{1}, \mathbf{2})$

and for $N_f = 3$:

Particle (n_m, n_e)	Rep. $SO(6) \cong SU(4)$
Quarks $(0, 1)$	$\mathbf{6}$
Dyons $(1, 2n + 1/2)$	$\mathbf{4}$
$(1, 2n + 3/2)$	$\bar{\mathbf{4}}$
$(2, 2n + 1)$	$\mathbf{1}$

The compactification $R_2 \to 0$ corresponds to the well-known limit of M-theory giving Type IIA string theory. The four-dimensional $\mathcal{N} = 2$, $SU(2)$ gauge theory is engineered by taking a double scaling limit in the Kähler moduli space of the non-compact Calabi-Yau [65]. Therefore, one can in principle arrive at this spectrum by continuation of semi-stable sheaves, which are the mathematical description of BPS-states in the large volume limit, to the field theory regime of the Kähler moduli space. The electric-magnetic and flavor charges are given in terms of the Chern classes by: $n_m = r$, $\mathbf{k} = c_1 + rK_{B_{N_f}} = \frac{1}{2} n_e \mathbf{f} - \frac{1}{2} \sum_i n_{f,i}\, \boldsymbol{d}_i$.

Note that there are no charges in the field theory corresponding to D0-brane (second Chern character) or D2-brane (first Chern character) supported on $\mathbf{C}$, since these objects become very massive in the field theory limit and leave the spectrum. Therefore, at most the lowest term in the q-expansion of the $\mathcal{N}=4$ Yang-Mills partition functions correspond to monopoles and dyons in the field theory. To determine which of the lowest term indeed represent BPS-states of the field theory, one has to verify that their mass is at the field theory scale and not of string scale, and that furthermore no walls of marginal stability are crossed by the Kähler moduli in between the large volume limit and the field theory limit. This analysis is carried out for pure $SU(2)$ gauge theory in [66]. In the following, we will discuss some qualitative features for $N_f > 0$ without performing the full analysis.

The correlation between electric and flavor charges [33, 34] can be seen from this perspective as a natural consequence of the gluing vectors of the lattice $A \oplus D$. The W-boson lies in the conjugacy class of $\boldsymbol{g}_0$, and the quarks in the conjugacy class of $\boldsymbol{g}_2$. The two other conjugacy classes $\boldsymbol{g}_{1,3}$ are not relevant for $r=0$ due to the large size of the curve $\mathbf{C}$. Since the charge vector $\mathbf{k}$ differs from the first Chern class c_1 by $rK_S/2$, the electric and flavor charges of monopoles with odd r take values in the classes $\boldsymbol{g}_{1,3}$. For N_f even, there is one sector with even n_e and one with odd; for N_f odd this is shifted by $\frac{1}{2}$. The dimension of the $SO(2N_f)$ representation of the monopoles and dyons are provided by the first coefficients of the theta functions (4.21). These indeed agree with the dimensions listed in the above tables.

To further verify that the dyons lie in hypermultiplets whose BPS index is 1, we need to consider the functions $h_{1,c_1}(z,\tau;\mathbb{B}_n)$. The generating functions for $r=1$ and any choice of c_1 have the expansion:

$$h_{1,c_1}(z,\tau;\mathbb{B}_n) = \frac{q^{-\frac{4+n}{24}}}{w-w^{-1}}\left(1+(w^{-2}+2+n+w^2)\,q+\dots\right), \tag{4.43}$$

which indeed confirms that the dyons lie in hypermultiplets.

With the results of Section 4.2, we can also address the monopole with magnetic charge 2 expected in the BPS-spectrum of SYM with $N_f=3$ [31, 33, 34]. Since r is even and the electric charge is odd, c_1 lies in the conjugacy class $\boldsymbol{g}_2$. The complex dimension of the moduli space of rank 2 sheaves is: $4c_2 - c_1^2 - 3$, which should vanish in order to have BPS index equal to 1. This can occur for $c_2 = 0$ only for $N_f = 3$, namely if $c_1 = \sum_{i=1}^3 \pm \mathbf{c}_i$. In order to further verify that a semi-stable sheaf exist we expand $h_{2,c_1}(z,\tau;\mathbb{B}_n, J_{\varepsilon,1})$. Expanding Eq. (4.39) gives:

$$h_{2,c_1}(z,\tau;\mathbb{B}_n,J_{\varepsilon,1}) = \frac{q^{\frac{1}{6}}}{w-w^{-1}}\left(1+(w^{-4}+6w^{-2}+16+6w^2+w^4)\,q+\dots\right). \tag{4.44}$$

Thus indeed a semi-stable sheaf exists with BPS invariant equal to 1. This state is a singlet in the field theory limit according to the table of BPS states for $N_f=3$. Thus, to further proof the existence of this bound state in the field theory limit, one has to show that a single linear combination of the $\mathbf{c}_i$ survives along a path in moduli space from large volume to the weak coupling chamber of the field theory limit.

5 Discussion and outlook

We have proposed a correspondence between $(0, 4)$ sigma models with target space the moduli spaces of r static monopoles in $SU(2)$ four-dimensional gauge theory, and $\mathcal{N} = 4$ $U(r)$ Yang-Mills theory on the four-manifolds known as del Pezzo surfaces. This correspondence can be understood from the point of view of geometric engineering of five-dimensional gauge theory by M-theory compactified on a non-compact Calabi-Yau manifold. The del Pezzo surfaces form the compact part of these Calabi-Yau manifolds. For $r = 1$, we have proven the correspondence, while for higher rank $r \geq 2$, much work remains to be done. Clearly, the computation of the 2d elliptic genus for $r > 2$ similarly to the 4d SYM computation is desirable. Another important missing point is the derivation of the sigma model of the monopole moduli space considered in this paper directly from the field theory obtained from the reduction of the degrees of freedom of multiple M5-branes wrapping $\mathbb{B}_0$ to two dimensions, along the lines of [3–5] for the case of a single fivebrane and compact Calabi-Yau threefolds.

We have left unexplored various aspects of the 5d/2d/4d correspondence. For example, to consider the more general class of four-manifolds which are $\mathbb{P}^1$ fibrations over a genus g Riemann surface instead of over a $\mathbb{P}^1$. This would lead in the engineered $SU(2)$ gauge theory to g hypermultiplets in the adjoint representation [67], whereas in our discussion we restricted to blow-ups of $\mathbb{P}^1 \otimes \mathbb{P}^1$, leading to hypermultiplets in the fundamental representation. Also, we have not included possible mass terms for the hypers in our analysis, apart from some rather simple observations made in Sections 2 and 3. Including them in the elliptic genus is an interesting extension. Higher rank gauge groups $SU(N > 2)$ in five-dimensional gauge theory can also be considered. They could lead to new versions of the 5d/2d/4d correspondence.

Another interesting direction for future research is the study of instanton effects of the three-dimensional theory obtained from the five-dimensional theory compactified on T^2 [22]. These corrections contribute to the hypermultiplet moduli space metric on the Coulomb branch of the three-dimensional effective action. The hypermultiplet moduli space was shown to be equal to the moduli space of doubly periodic monopoles in [68], but the metric remains difficult to be computed. In [22], it was argued that for $N_f = 0$, the elliptic genus determines the instanton induced four-fermi correlator in the three-dimensional gauge theory, and hence the hypermultiplet moduli space metric. Furthermore, these corrections are beautifully captured using integrals over twistor space [64, 69, 70], and are expected to be invariant under $SL_2(\mathbb{Z})$ transformations of the T^2. Progress on these aspects for one-instanton corrections is recently made in [71]. An intriguing interplay is expected for higher instanton corrections between the period integrals as in Eq. (4.42) and twistor integrals.

Finally, it would be worth investigating five-dimensional gauge theories, possibly at the superconformal point, on different manifolds, such as $S^3 \times T^2$, to see if a 5d/2d/4d correspondence still holds. If so, connections could be made to the study of partition functions of five-dimensional gauge theories using localization techniques, along the lines of [72–76]. We leave this for future investigation.

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